# Entanglement of random pure states of bipartite indistinguishable systems

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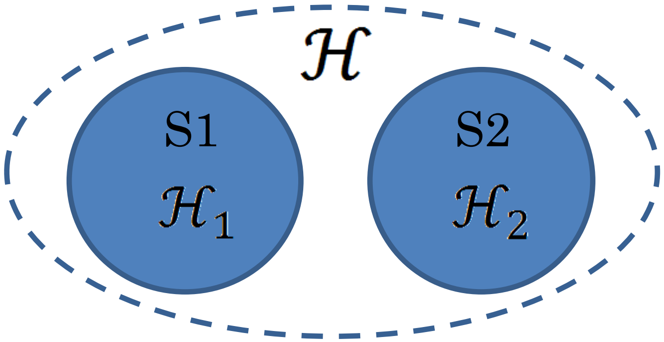
# Outline

- Entanglement
- Identical and indistinguishable particles
- Entanglement of indistinguishable particles
- Random states

# ENTANGLEMENT - I




$$\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2 \qquad \sigma_a = \sigma \cdot \hat{a}$$

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \qquad \sigma_b = \sigma \cdot \hat{b}$$

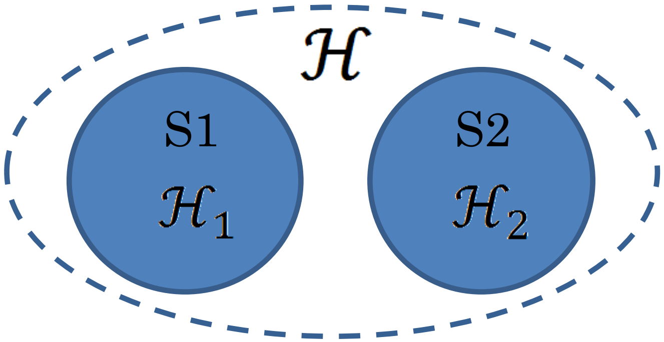
Product State

$$|\psi\rangle = |\varphi\rangle_1 |\varphi\rangle_2$$

$$\rho = \rho_1 \otimes \rho_2$$

$$\langle\psi|\sigma_a \otimes \sigma_b|\psi\rangle = \underbrace{\langle\varphi|\sigma_a|\varphi\rangle_1 \langle\varphi|\sigma_b|\varphi\rangle_2}_{\text{Sub-Systems are uncorrelated}}$$

# Entanglement - I

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Sub-Systems are uncorrelated

Entangled State

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

$$\rho \neq \rho_1 \otimes \rho_2$$

$$\langle\psi|\sigma_a \otimes \sigma_b|\psi\rangle = \underbrace{-\cos\theta_{ab}}$$

Sub-Systems are correlated

Entangled states are those that are not product states

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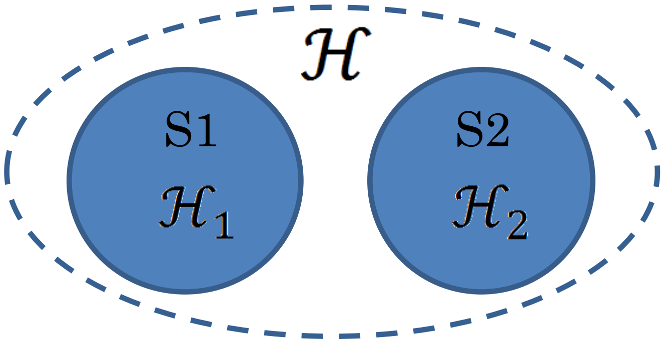


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Sub-Systems are correlated

Entangled states are those that are not product states
Entanglement depends on the tensor product structure of the system

# Entanglement - II

**How to know if a given bipartite state is entangled or not?**

$$|\psi\rangle = \sum_{i,j=1}^{n_i n_j} c_{ij} \, |\varphi_i\rangle |\phi_j\rangle \xrightarrow[\text{Change of Basis}]{\text{Schmidt decomposition}} |\psi\rangle = \sum_{i=1}^{n_i} \lambda_i \, \underbrace{|\tilde{\varphi}_i\rangle |\tilde{\phi}_i\rangle}_{\text{Product States}}$$

$\lambda_i$ → Schmidt Coefficients

If there is more than one Schmidt coefficient different from 0 the state is entangled

# ENTANGLEMENT - III

**How to quantify entanglement ?**

LOCC = Local Operations and Classical Communication

$$u_1 \otimes u_2|\psi\rangle = \sum_{i,j}^{n_j n_i} C_{ij} u_1 \otimes u_2 |\varphi_i\rangle|\phi_j\rangle$$

Only classical correlations

**Some Measures**

$$E[|\psi\rangle] = -Tra[\rho_r Ln(\rho_r)] = -\sum_{i}^{min(n_j n_i)} \lambda_i \ln \lambda_i$$

Entropy

$$C[|\psi\rangle] = |\langle\psi^*|\sigma_y \otimes \sigma_y|\psi\rangle|$$

Concurrence

Under LOCC conditions, Schmidt coefficients contains all the information about the entanglement of the system

# Identical and indistinguishable particles

Symmetrization postulate

Fermions, Half –integer spin, anti-symmetric states

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle_1|\beta\rangle_2 - |\beta\rangle_2|\alpha\rangle_1)$$

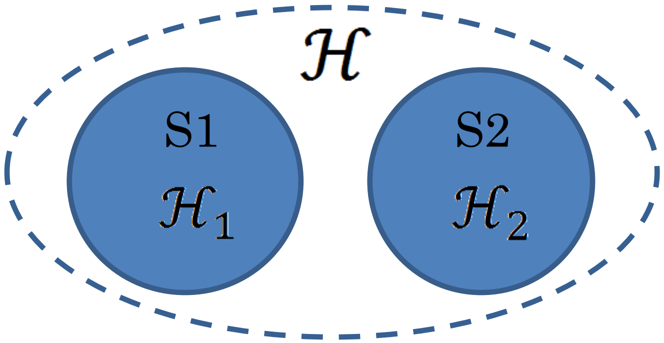
Bosons, integer spin, symmetric states

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_2|\alpha\rangle_1)$$

Are these states entangled?

Which is the nature of entanglement on identical particle systems?

# IDENTICAL AND INDISTINGUISHABLE PARTICLES




$$\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^n$$

Symmetrization postulate ⇒ No (obvious) tensor product structure for the composite system

| **Distinguishable particles** | **Fermions** | **Bosons** |
|---|---|---|
| $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$ | $\mathcal{H} = \mathcal{A}[\mathbb{C}^n \otimes \mathbb{C}^n]$ | $\mathcal{H} = \mathcal{S}[\mathbb{C}^n \otimes \mathbb{C}^n]$ |
| $n^2$ | $\frac{(n-1)n}{2}$ | $\frac{n(n+1)}{2}$ |

Dimension of the Composite Hilbert Space

# Indistinguishable Particles Entanglement of Particles-I

**Fermions**

Change of Basis →

$$|\psi\rangle = \sum_{i,j=1}^{K} w_{ij}\, f_i^\dagger f_j^\dagger |0\rangle \qquad |\psi\rangle = \sum_{i=0}^{[K/2]-1} F_i\, \tilde{f}_{2i+1}^\dagger \tilde{f}_{2(i+1)}^\dagger |0\rangle$$

Block-Diagonal → $F = uwu^T$

**Bosons**

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Diagonal → $B = uzu^T$

John Schliemann, et al. PRL. A 64, 022303 (2001)
R.Paskauskas, L.You. PRl.A 64, 042310 (2001)

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# Entanglement of Particles-I

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Is this formal analogy enough?


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R.Paskauskas, L.You. PRl.A 64, 042310 (2001)

# Bipartite random states of fermionic and bosonic systems

Why random states?

- Entanglement is an useful resource for quantum computation and randomness is a way to create it.
- Useful in super dense coding, remote state preparation, data hiding protocols.
- They provide a natural benchmark for assessing: schemes to distinguish quantum states.

Random states allow to asset general behaviors with minimal prior information.

# Bipartite random states of fermionic and bosonic systems

The systems of our interest:

| | | | | |
|---|---|---|---|---|
| Distinguishable Particles | (2Q-bits) | $\mathcal{H}s=C^2$ | $\mathcal{H}=\mathcal{H}s\otimes\mathcal{H}s$ | $\mathrm{Dim}(\mathcal{H})=4$ |
| 2 Fermions 4 Levels each | | $\mathcal{H}s=C^4$ | $\mathcal{H}=\mathcal{A}(\mathcal{H}s\otimes\mathcal{H}s)$ | $\mathrm{Dim}(\mathcal{H})=6$ |
| 2 Bosons 2 Levels each | | $\mathcal{H}s=C^2$ | $\mathcal{H}=S(\mathcal{H}s\otimes\mathcal{H}s)$ | $\mathrm{Dim}(\mathcal{H})=3$ |

# How do we quantify indistinguishable particles entanglement?

CONCURRENCE:

$$
c[|\psi\rangle] = \begin{cases} \text{Distinguishable particles} \\ 2|c_{12}c_{21} - c_{22}c_{11}| \\ \text{Fermions} \\ 8|w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23}| \\ \text{Bosons} \\ 4|z_{11}z_{22} - z_{12}^2| \end{cases}
$$

K.Eckert, et al. Annals of Physics 299, 88-127(2002)

# Bipartite random states of fermionic and bosonic systems

Which distribution of states?

Distribution for fermions

$$P_f[|\psi\rangle] = \frac{15}{16\pi^6}\delta\left(\frac{1}{4} - \sum_{i<j}^{4} |w_{ij}|^2\right)$$

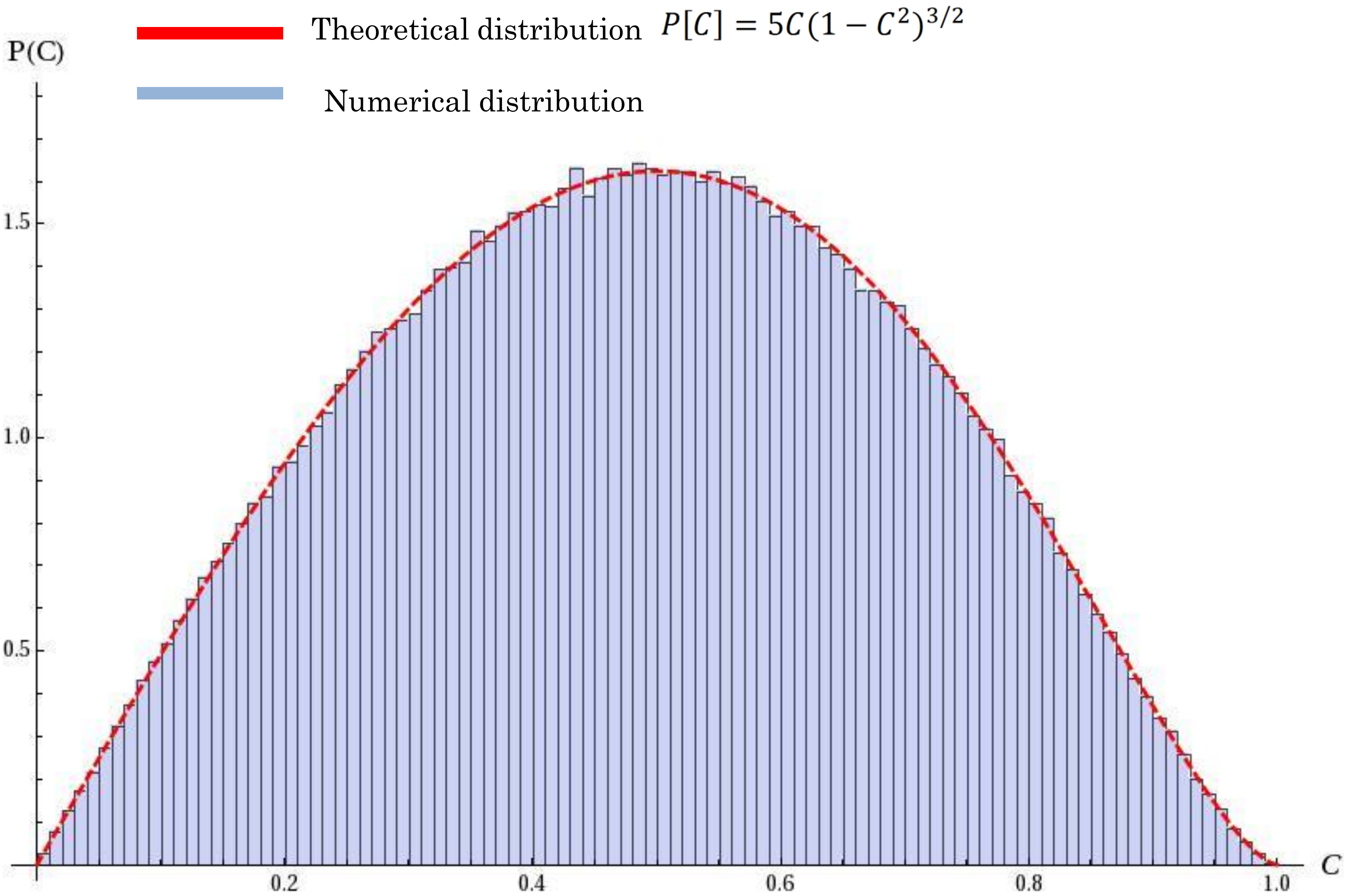
Distribution for bosons

$$P_b[|\psi\rangle] = \frac{1}{4\pi}\delta\left(1 - \sum_{i=1}^{3} |\alpha_i|^2\right)$$

We are interested in the behavior of concurrence for random states.

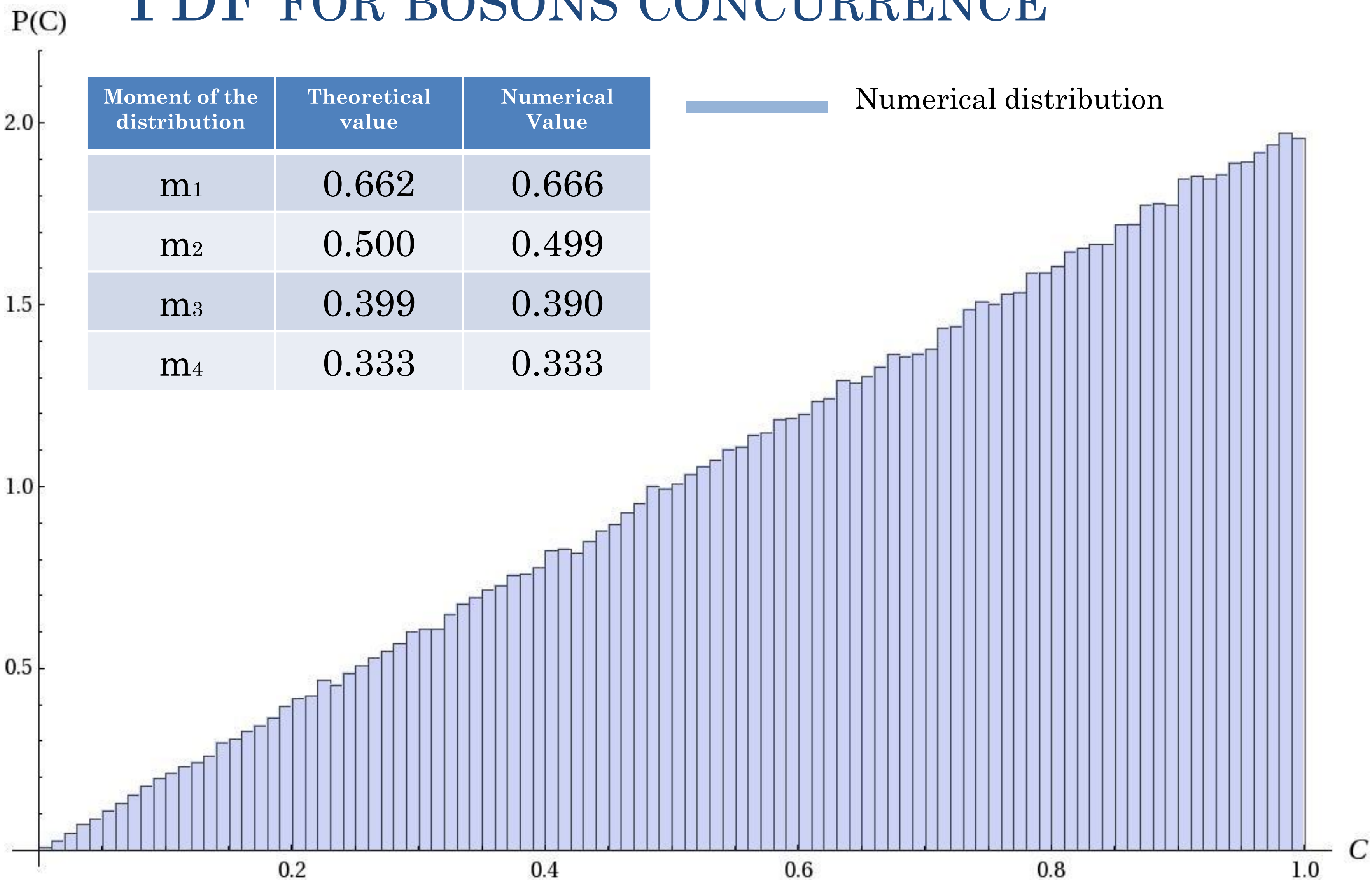
$$P(c) = \int [d|\psi\rangle]\, \delta\big(c - c(|\psi\rangle)\big) P(|\psi\rangle)$$

# PDF FOR FERMIONS CONCURRENCE



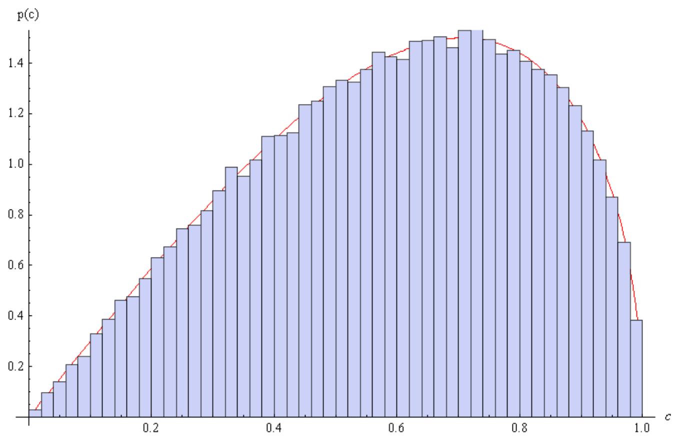
# PDF FOR BOSONS CONCURRENCE

| Moment of the distribution | Theoretical value | Numerical Value |
|---|---|---|
| $m_1$ | 0.662 | 0.666 |
| $m_2$ | 0.500 | 0.499 |
| $m_3$ | 0.399 | 0.390 |
| $m_4$ | 0.333 | 0.333 |

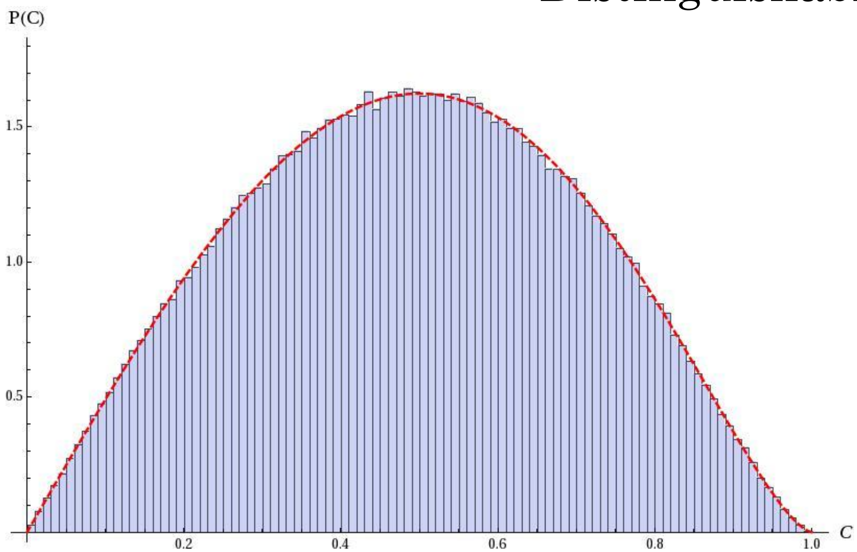



$$m_n = \frac{1}{4\pi^3}\int \left(Cos^2(2\omega) + Sin^2(2\omega)F^2(\Omega_1,\Omega_2)\right)^{n/2} Sin^2(2\omega)\, d\omega\, d\Omega_1\, d\Omega_2$$

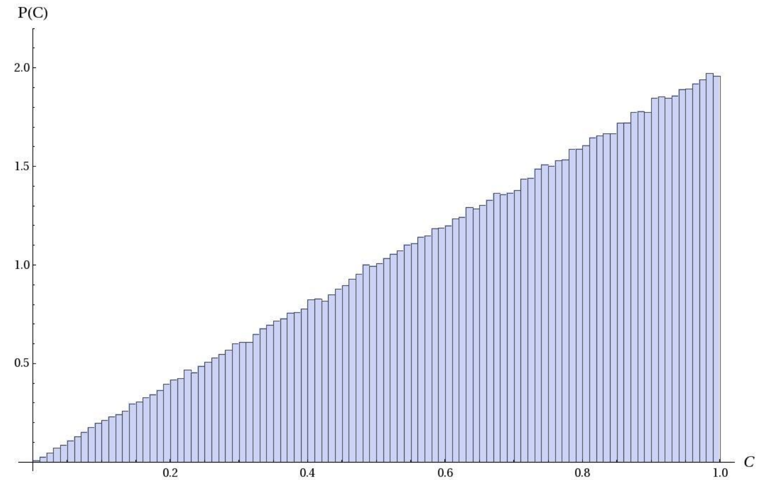
# PDF for distinguishable particles, Fermions and bosons




Distinguishable particles




Fermions




Bosons

# Conclusions and Outlook

How is the behavior of the distributions for higher dimensions?

What happens for multi-particle systems? (E.g Inequivalent Entanglement properties W ,GHZ states)

Is the LOCC paradigm important for quantum correlations in systems of identical particles?

# Thank You

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