Phase-space approach to polarization

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Cooperations

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Outline

- Motivation
- Stokes parameters and Stokes operators
- Degree of polarization
- Basic tools in quantum tomography
- Tomography of polarization states
- Experimental results
- Conclusions

Motivation

- Advantages of polarization states

  - ✓ Robust
  - ✓ Simple to transform
  - ✓ Only marginal losses
  - ✓ Efficiently measured

Polarization is an excellent candidate to encode quantum information

- Photon counting detectors are used to measure the polarization: The post-selected polarization states are number states

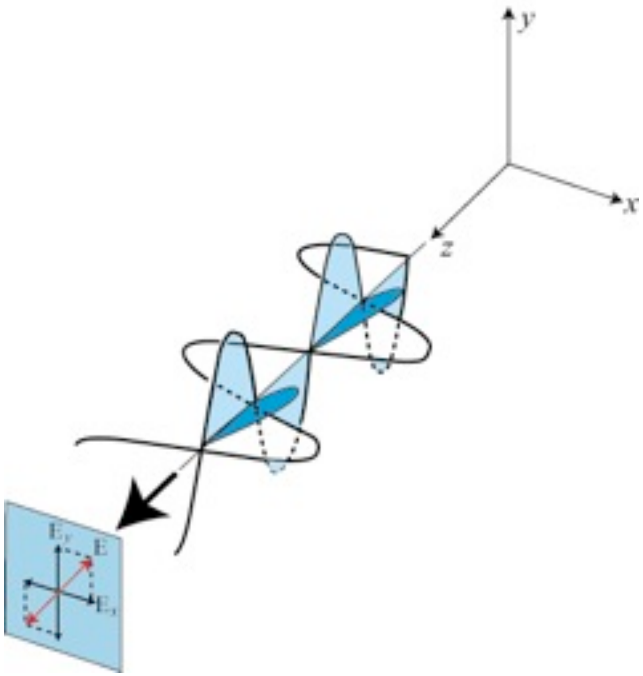
A (semi)classical description of polarization is insufficient

Classical description of polarization

Monochromatic plane wave in a linear, homogeneous, isotropic medium

$$\mathbf{E}(z,t) = \mathbf{E}_0 \exp[-i(\omega t - kz)]$$

$\mathbf{E}_0$ is a complex vector that characterizes the state of polarization



$$\mathbf{E}_0 = a_H \mathbf{e}_H + a_V \mathbf{e}_V$$

linear-polarization basis: $(\mathbf{e}_H, \mathbf{e}_V)$

circular-polarization basis: $(\mathbf{e}_+, \mathbf{e}_-)$

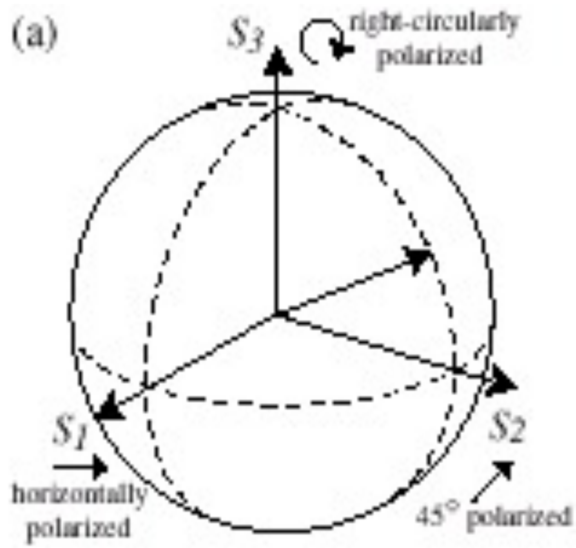
$$\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_H \pm i\mathbf{e}_V)$$

Stokes parameters and Poincaré sphere

$$
\begin{aligned}
S_0 &= a_H^* a_H + a_V^* a_V \\
S_x &= a_H a_V^* + a_H^* a_V \\
S_y &= i(a_H a_V^* - a_H^* a_V) \\
S_z &= a_H^* a_H - a_V^* a_V
\end{aligned}
$$

$$s_x = \frac{S_x}{S_0} \qquad s_y = \frac{S_y}{S_0} \qquad s_z = \frac{S_z}{S_0}$$

$$s_x^2 + s_y^2 + s_z^2 = 1$$



Transformations on the Poincaré sphere

- (Linear) polarization transformations

$$\begin{pmatrix} a'_H \\ a'_V \end{pmatrix} = \mathsf{U} \begin{pmatrix} a_H \\ a_V \end{pmatrix}, \qquad \mathsf{U} \in \mathsf{SU(2)}$$

- Corresponding transformations in the Poincaré sphere

$$\begin{pmatrix} s'_1 \\ s'_2 \\ s'_3 \end{pmatrix} = \mathsf{R(U)} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \qquad \mathsf{R(U)} \in \mathsf{SO(3)}$$

Transformations on the Poincaré sphere

Examples

$$\begin{pmatrix} e^{-i\varphi_H} & 0 \\ 0 & e^{-i\varphi_V} \end{pmatrix} \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A differential phase shift induces a rotation about Z

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \mapsto \begin{pmatrix} \cos(2\vartheta) & 0 & \sin(2\vartheta) \\ 0 & 1 & 0 \\ -\sin(2\vartheta) & 0 & \cos(2\vartheta) \end{pmatrix}$$

A geometrical rotation of angle θ induces a rotation about Y of angle 2θ

Quantum fields

- Quantum version: Replace classical amplitudes by bosonic operators

$$[\hat{a}_j^\dagger, \hat{a}_k] = \delta_{jk}\,, \qquad j,k \in \{H, L\}$$

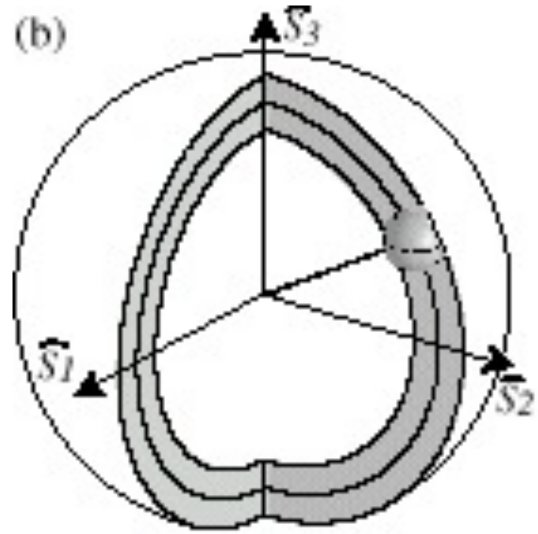
- Stokes parameters appear as average values of Stokes operators

$$\hat{\varrho} = \frac{1}{2}(\hat{\mathbb{1}} + \mathbf{s}\cdot\boldsymbol{\sigma})\,, \qquad \mathbf{s} = \mathrm{Tr}(\hat{\varrho}\boldsymbol{\sigma})$$

- **s** is the polarization (Bloch) vector

$$\langle \Delta \hat{s}_x^2 \rangle + \langle \Delta \hat{s}_y^2 \rangle + \langle \Delta \hat{s}_z^2 \rangle \geq 2 \langle \hat{S}_0 \rangle$$

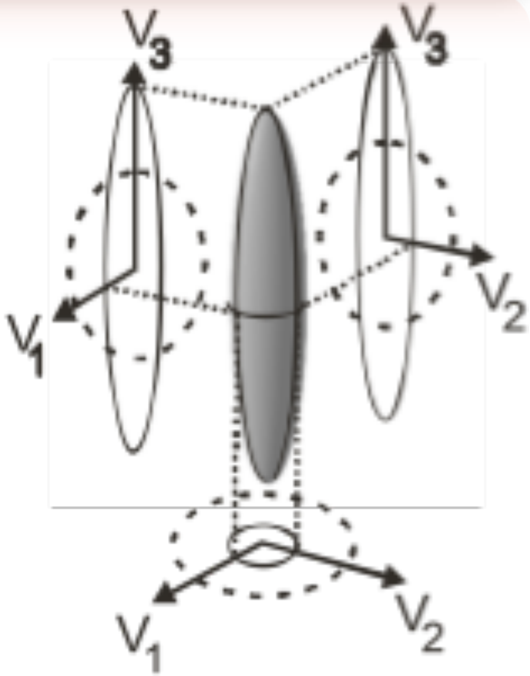
The electric field vector does not describe a definite ellipse!



Polarization squeezed states

Heisenberg uncertainty relation

$$\Delta^2 S_i \, \Delta^2 S_j \geq |\langle S_k \rangle|^2$$



Polarisation Squeezed State:

$$\Delta^2 S_i < |\langle S_k \rangle|^2 < \Delta^2 S_j$$

$$\Downarrow$$

State dependent!

W. P. Bowen *et al*, Phys. Rev. Lett. **88**, 093601 (2002)
J. Heersink *et al*, Phys. Rev. A **68**, 013815 (2003)
N. Korolkova *et al*, Phys. Rev. A **65**, 052306 (2002)

Classical degree of polarization

Semiclassical definition

$$\mathbb{P}_1 = \frac{\sqrt{\langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2 + \langle \hat{S}_z \rangle^2}}{\langle \hat{S}_0 \rangle}$$

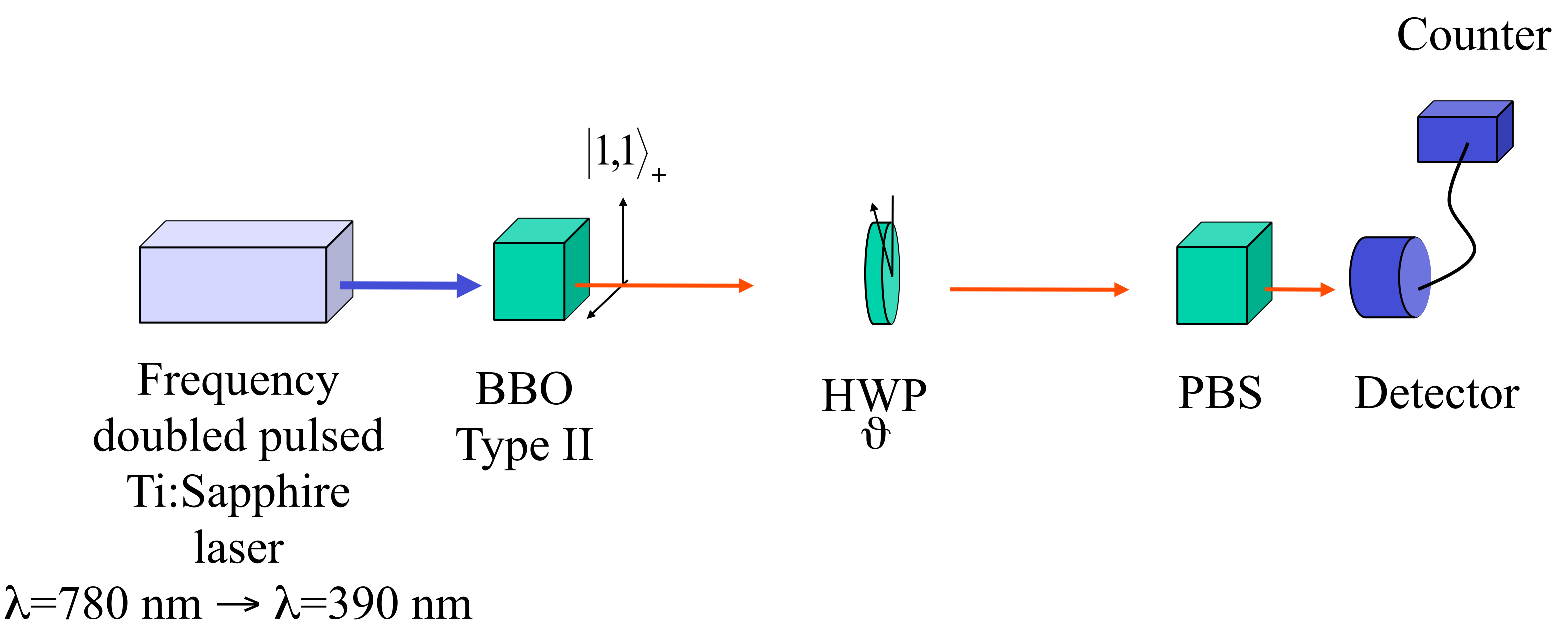
Problems

- ✓ It is defined solely in terms of the first moment of the Stokes operators.
- ✓ There are states with P=0 that cannot be regarded as unpolarized.
- ✓ P does not reflect the lack of perfect polarization for any quantum state.
- ✓ $P = 1$ for SU(2) coherent states (and this includes the two-mode vacuum) and any two-mode state $|\psi_H, 0_v\rangle$

Hidden polarization

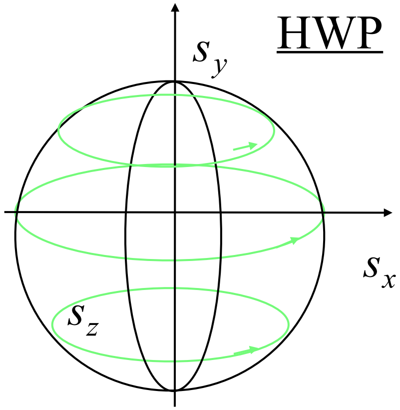
$\mathbb{P}_1 = 0 \Rightarrow$ Is the corresponding state is unpolarized?

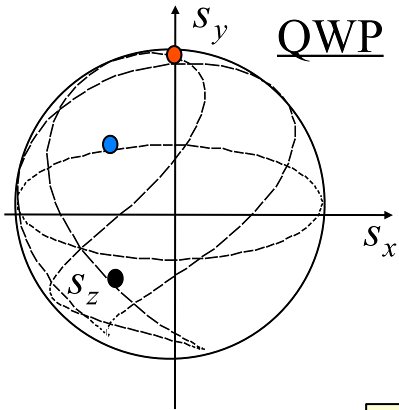
Consider the state $|1,1\rangle_+$

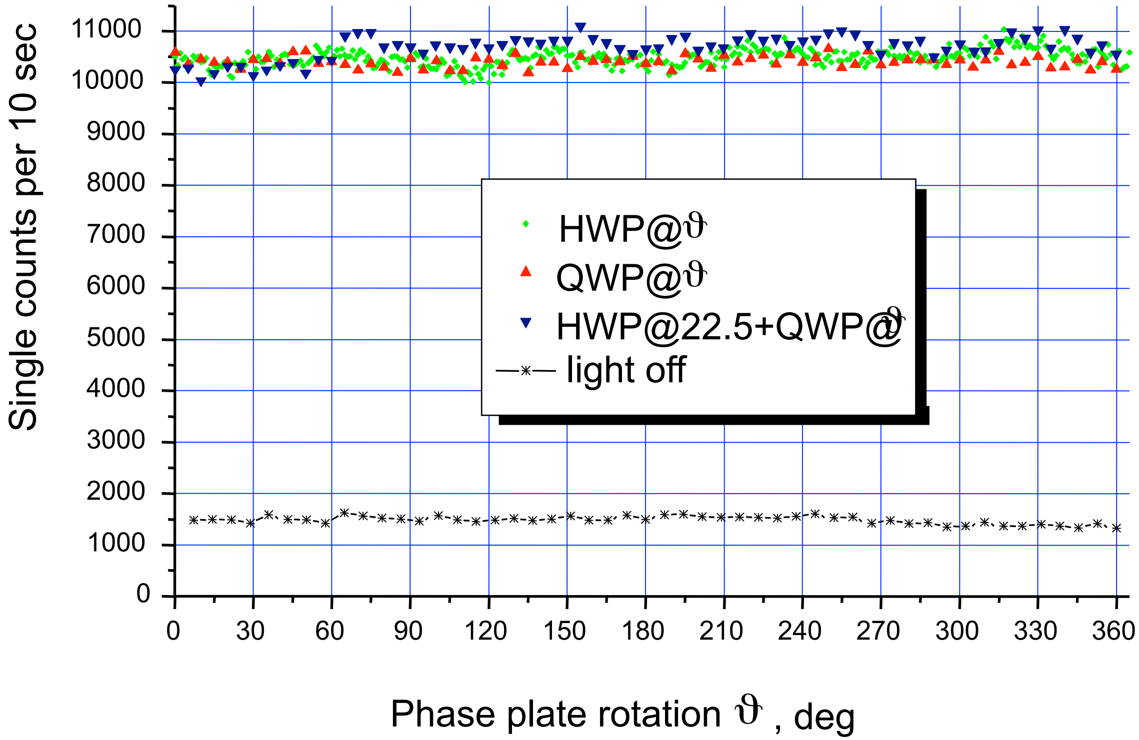


P. Usachev, J. Söderholm, G. Björk, and A. Trifonov, Opt. Commun. **193**, 161 (2001).
T. Tsegaye, et al. Phys. Rev. Lett., **85**, 5013 (2000.)

Experimental results



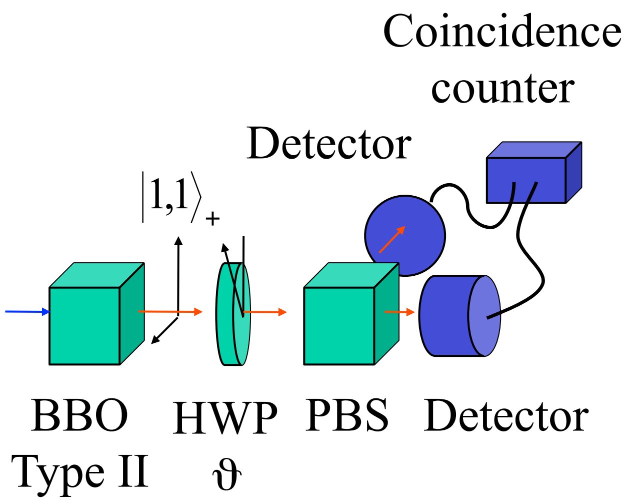


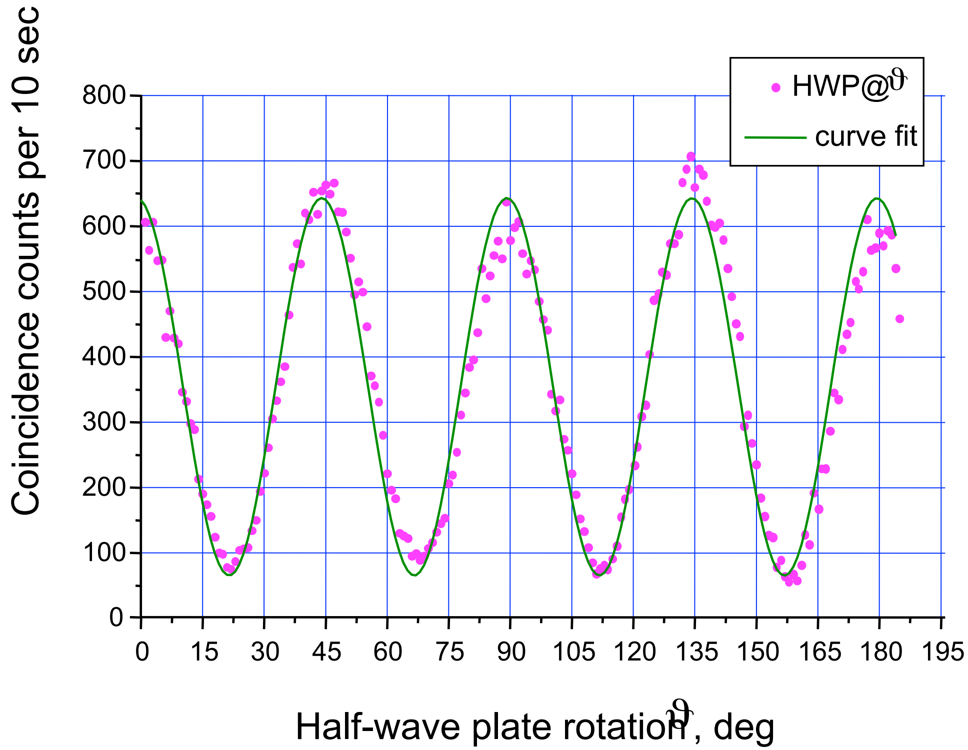


The state is unpolarized according to the classical definition

P. Usachev, J. Söderholm, G. Björk, and A. Trifonov, Opt. Commun. **193**, 161 (2001).

A coincidence experiment





Since the state is not invariant under geometrical rotation, it is not unpolarized. The raw data coincidence count visibility is ~ 76%, so the state has a rather high degree of (quantum) polarization although by the classical definition the state is unpolarized. This is referred to as "hidden" polarization.

D. M. Klyshko, Phys. Lett. A **163**, 349 (1992).

A second-order degree of polarization

- Characteristic properties
  - Scalar
  - Invariant under rotations on the Poincaré sphere

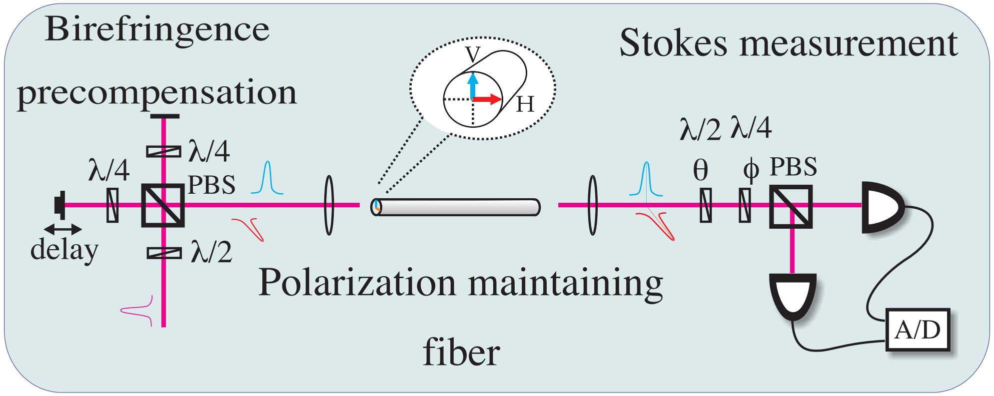
$$\mathbb{P}_2(\hat{\varrho}) = \sqrt{1 - \inf_{\mathbf{n}} \frac{(\Delta S_{\mathbf{n}})^2}{\frac{1}{3}\langle \hat{\mathbf{S}}^2 \rangle}}$$

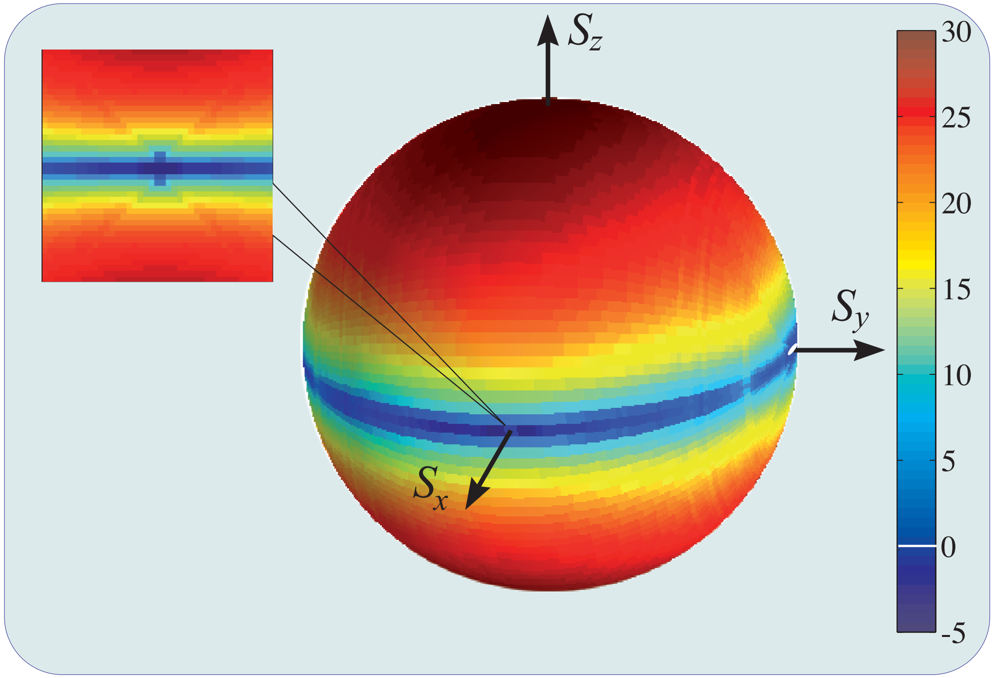
$$\Gamma_{k\ell} = \frac{1}{2}\langle \{ \hat{S}_k, \hat{S}_\ell \} \rangle - \langle \hat{S}_k \rangle \langle \hat{S}_\ell \rangle \qquad (\Delta S_{\mathbf{n}})^2 = \mathbf{n}^t \, \mathbf{\Gamma} \, \mathbf{n}$$

The minimum are the eigenvalues of the covariance matrix

A. B. Klimov et al., Phys. Rev. Lett.105, 153602 (2010)
L. L. Sanchez-Soto et al. Nature 353, 631 (2011)

Experimental results

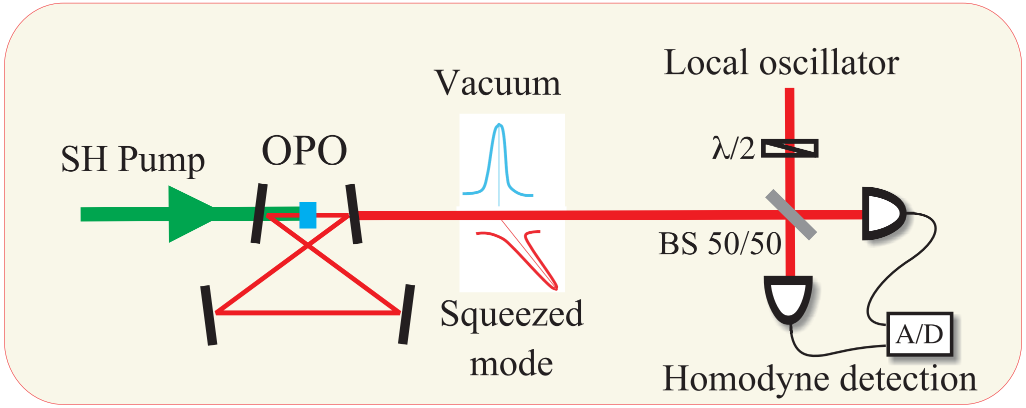


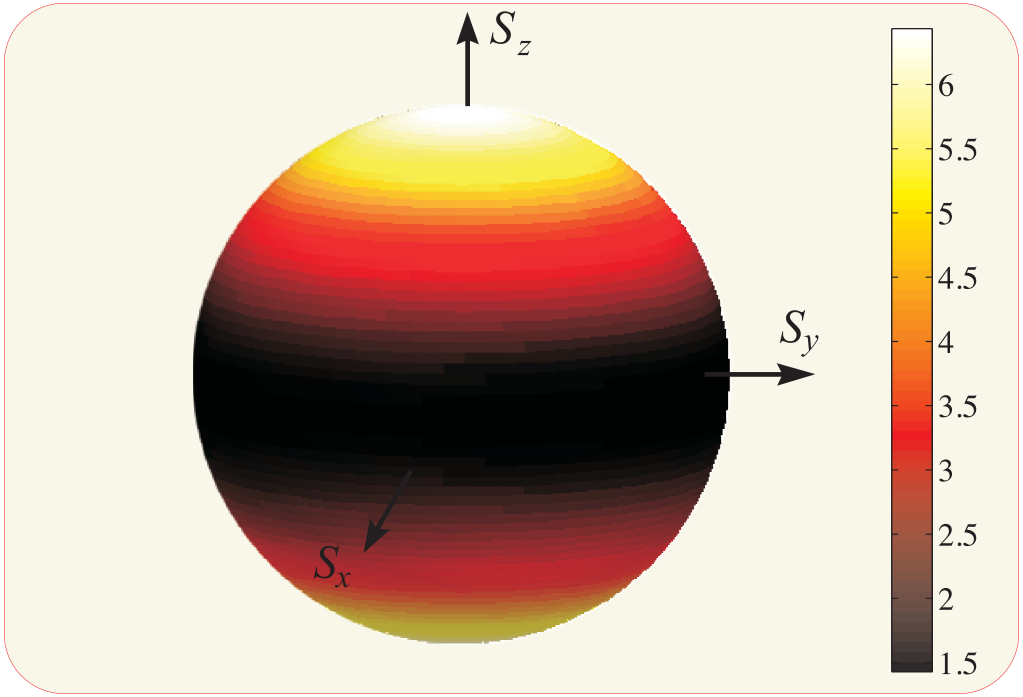


$$\mathbb{P}_1 = 1$$

$$\mathbb{P}_2 \simeq 1$$

Experimental results





$$\mathbb{P}_1 = 0.998 \pm 0.001$$
$$\mathbb{P}_2 = 0.79 \pm 0.01$$

Quantum tomography

measure different projections of several copies of a state → reconstruct the state

D. T. Smithey *et al*, Phys. Rev. Lett. **70**, 1244 (1993)

G. Breitenbach *et al* , Nature **387**, 471 (1997)

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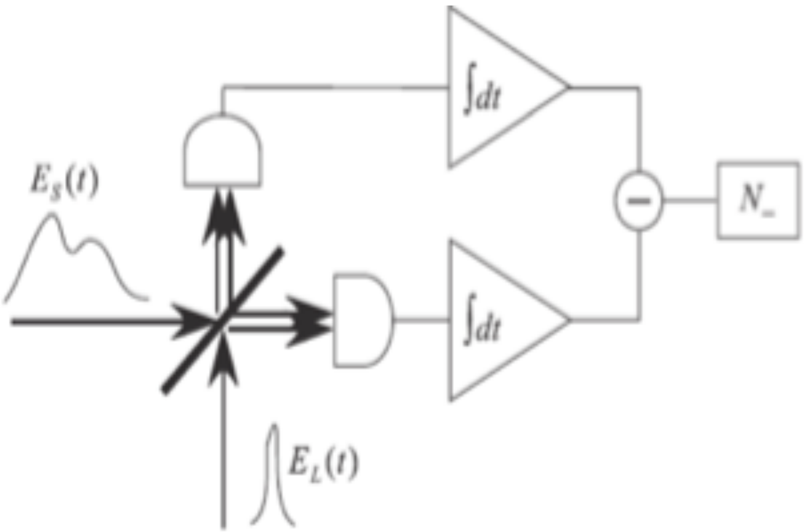

$$Q_\theta = X\cos\theta + P\sin\theta$$

Quantum tomography

measure different projections of several copies of a state → **reconstruct the state**

D. T. Smithey *et al*, Phys. Rev. Lett. **70**, 1244 (1993)

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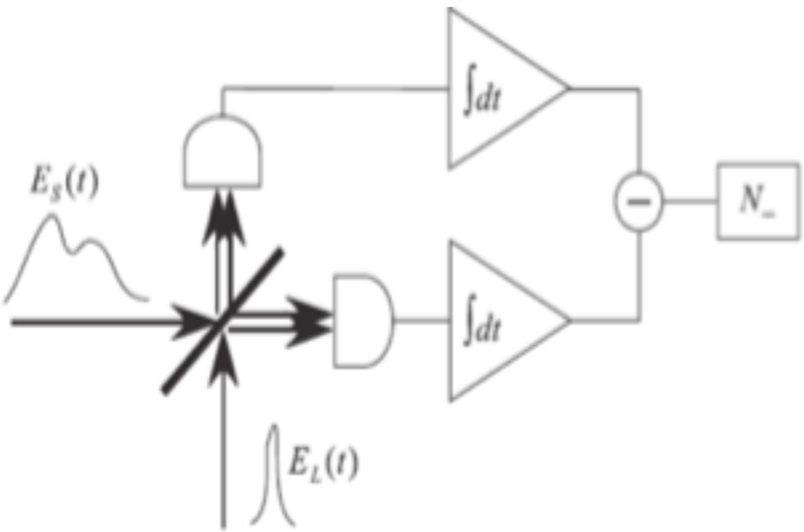
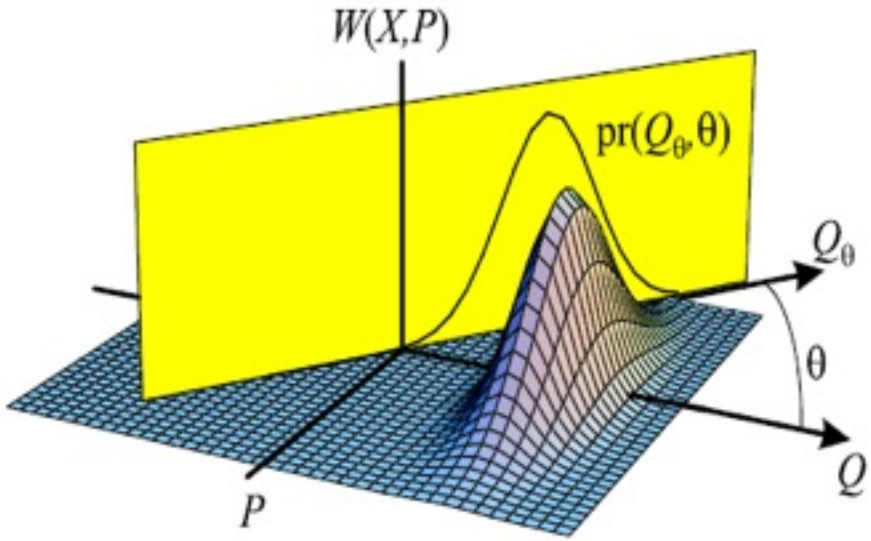




$$Q_\theta = X\cos\theta + P\sin\theta$$

$$W(X,P) = \frac{1}{2\pi^2}\int_0^\pi \int_{\mathbb{R}} w(Q_\theta,\theta) \times \mathcal{K}(X\cos\theta + P\sin\theta - Q_\theta)\,d\theta\,dQ_\theta$$

Key ingredients for quantum tomography

- Tomographically complete measurement
- Suitable representation of quantum states
- Robust algorithm to invert the experimental data

Polarization and spin

Elementary su(2) algebra

$$|J, m\rangle \equiv |n_H = J + m, n_V = J - m\rangle$$

The states $|J, m\rangle$ span an irreducible representation of SU(2)

$$J = \frac{1}{2}(n_H + n_V) \qquad m = \frac{1}{2}(n_H - n_V)$$

Detection POVM

$$\hat{\Pi}_{Jm} = |J, m\rangle\langle J, m|$$

The polarization sector: multipole expansion

Good news! We do not need to reconstruct all the density matrix!

$$\hat{\varrho}_{\mathrm{pol}} = \bigoplus_J \hat{\varrho}^{(J)}$$

$$\hat{\varrho}^{(J)} = \sum_{K=0}^{2J} \sum_{q=-K}^{K} \varrho_{Kq}^{(J)} \hat{T}_{Kq}^{(J)}$$

State multipoles

$$\varrho_{Kq}^{(J)} = \mathrm{Tr}[\hat{\varrho}^{(J)} \hat{T}_{Kq}^{(J)\,\dagger}]$$

Irreducible tensors

$$\hat{T}_{00}^{(J)} = \frac{1}{\sqrt{2J+1}} \hat{I}$$

$$\hat{T}_{1q}^{(J)} = \sqrt{\frac{3}{(2J+1)(J+1)J}}\, \hat{J}_q$$

The inversion

Measurable moments

$$\hat{J}_{\mathbf{n}} = \mathbf{n} \cdot \hat{\mathbf{J}} \qquad J^{\ell}_{\mathbf{n}}(\vartheta, \varphi) = \mathrm{Tr}[\hat{J}^{\ell}_{\mathbf{n}} \, \hat{\varrho}^{(J)}]$$

Useful result

$$J^{\ell}_{\mathbf{n}}(\theta, \phi) = \sqrt{\frac{4\pi}{2J+1}} \sum_{K=0}^{\ell} \sum_{q=-K}^{K} \varrho^{(J)}_{Kq} \, f^{(J)}_{K\ell} \, Y_{Kq}(\theta, \phi) \, .$$

The *l*-th moment contains information of all the state multipoles up to order *l*

The "redundant" inversion

Getting all the state multipoles

$$\varrho_{Kq}^{(J)} = \frac{1}{f_{K\ell}^{(J)}} \sqrt{\frac{2J+1}{4\pi}} \int_{S^2} d\Omega \, J_{\mathbf{n}}^{\ell}(\theta,\phi) \, Y_{Kq}^{*}(\theta.\phi)$$

Mapping the state on the Poincaré space via Wigner function

$$W^{(J)}(\theta,\phi) = \sum_{K=0}^{2J} \sum_{q=-K}^{K} \varrho_{Kq}^{(J)} \, Y_{Kq}^{*}(\theta,\phi)$$

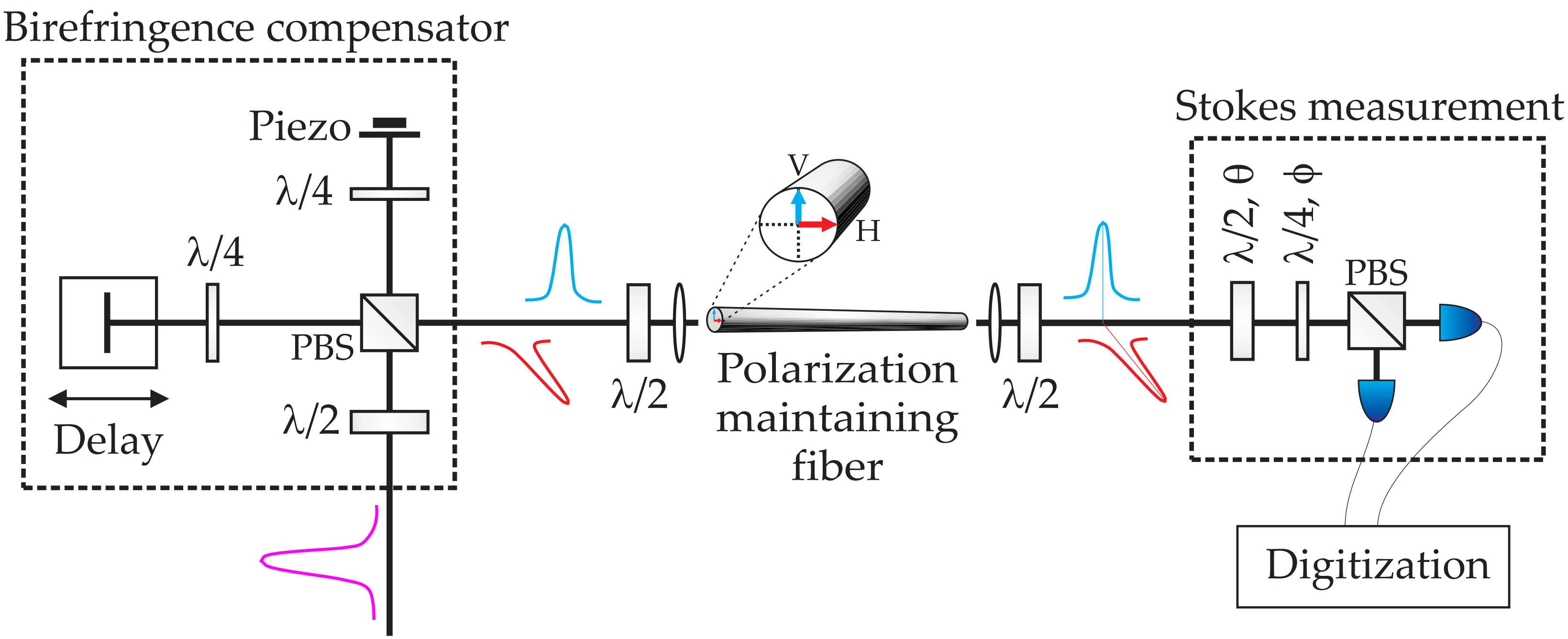
Limit of bright states

$$W(J,\theta,\phi) = \frac{2J+1}{4\pi^2} \int_{-\infty}^{\infty} dm \int_{S_2} d\mathbf{n}' \, \frac{d^2 w_m^{(J)}(\mathbf{n})}{dm^2} \, \delta(m - J\,\mathbf{n}\cdot\mathbf{n}')$$

Ch Müller et al., New J. Phys.. **14**, 220401 (2012)

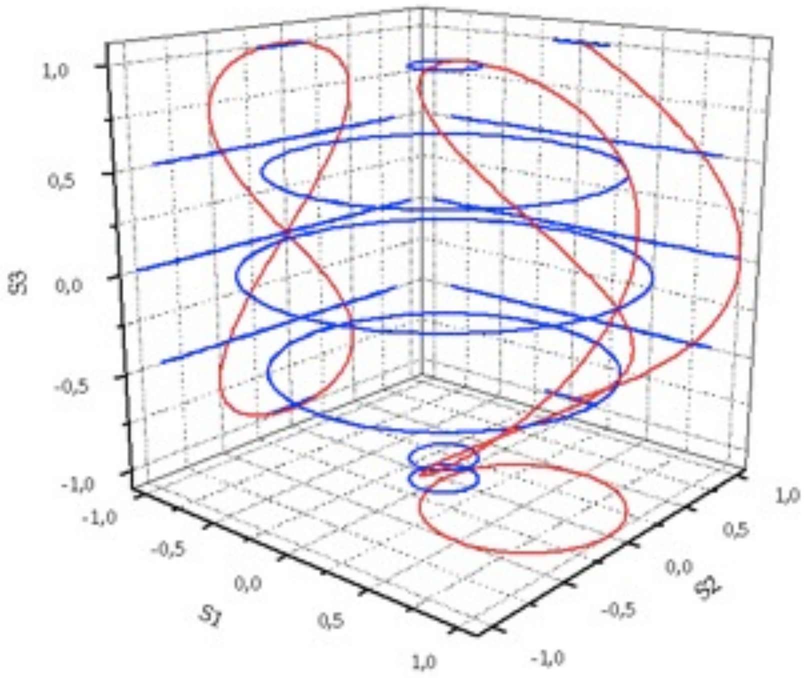
Inverse 3D Radon transform!

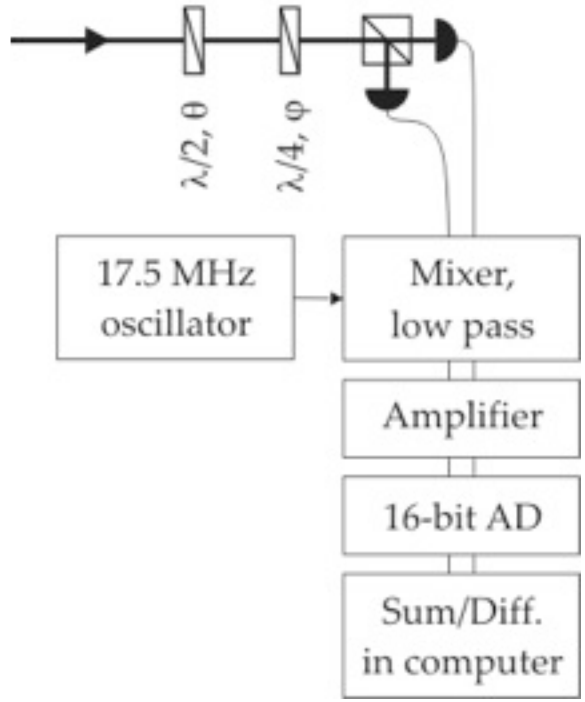
Experimental results



A. B. Klimov et al., Phys. Rev. Lett. **99**, 220401 (2007)

Stokes measurement





Different projections on the Poincaré sphere

The real measurement

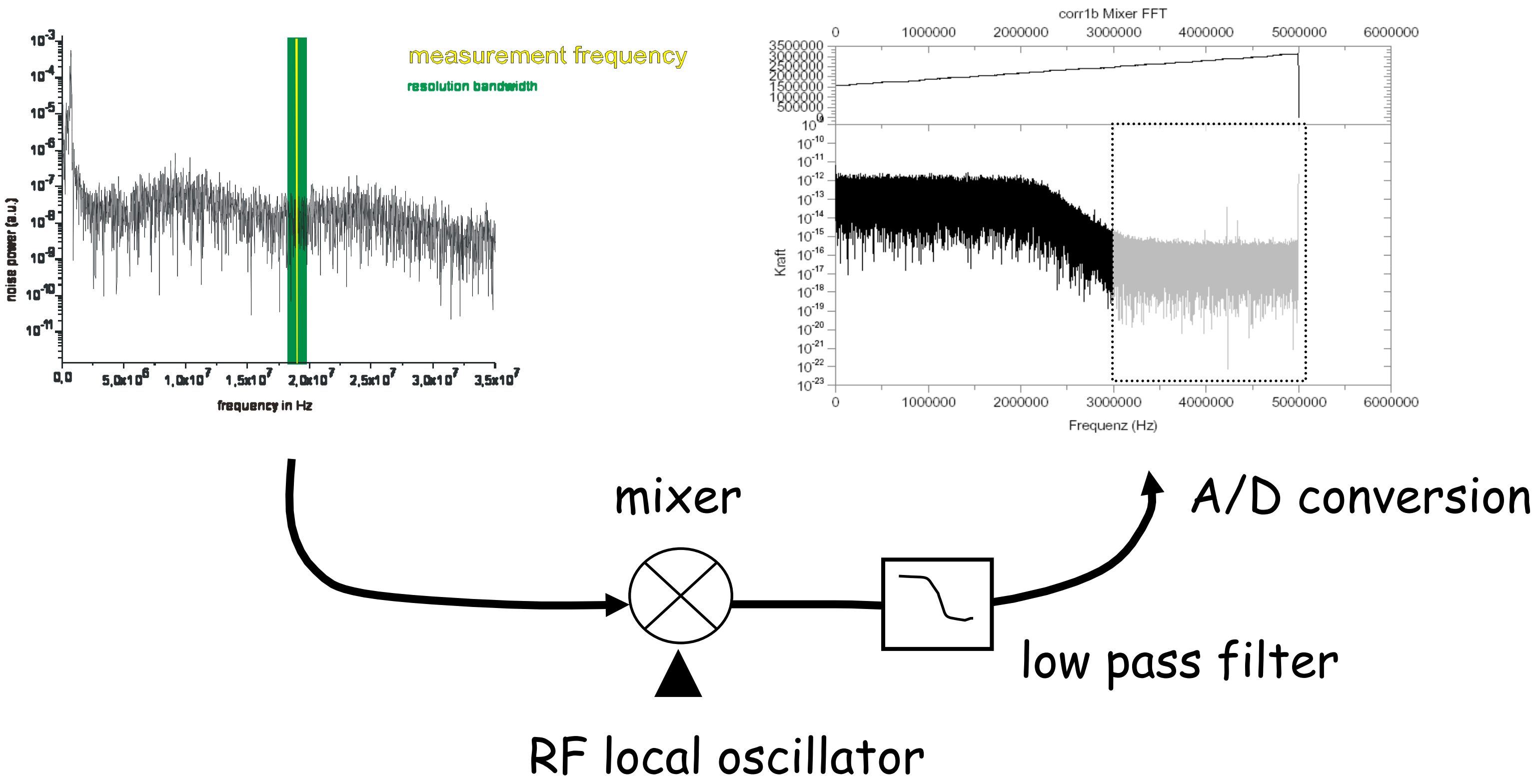
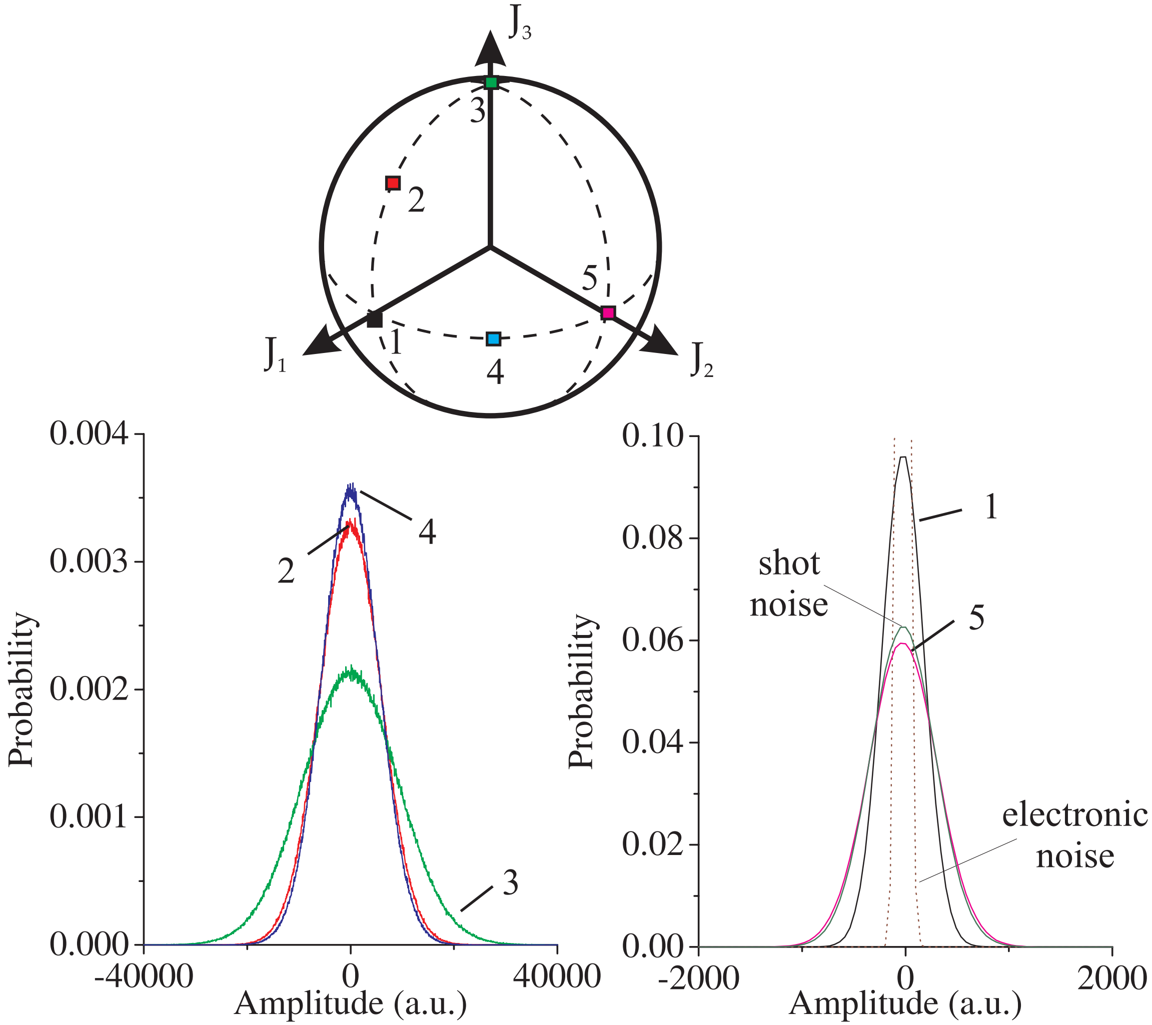


Photo current around measurement frequency (17.5 MHz)

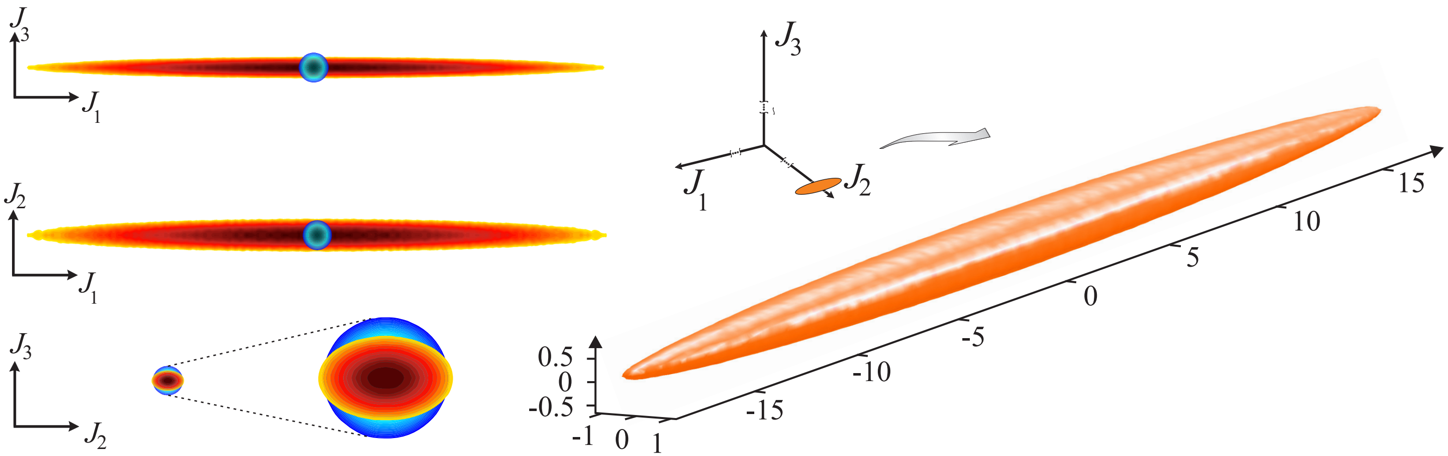
Experimental results



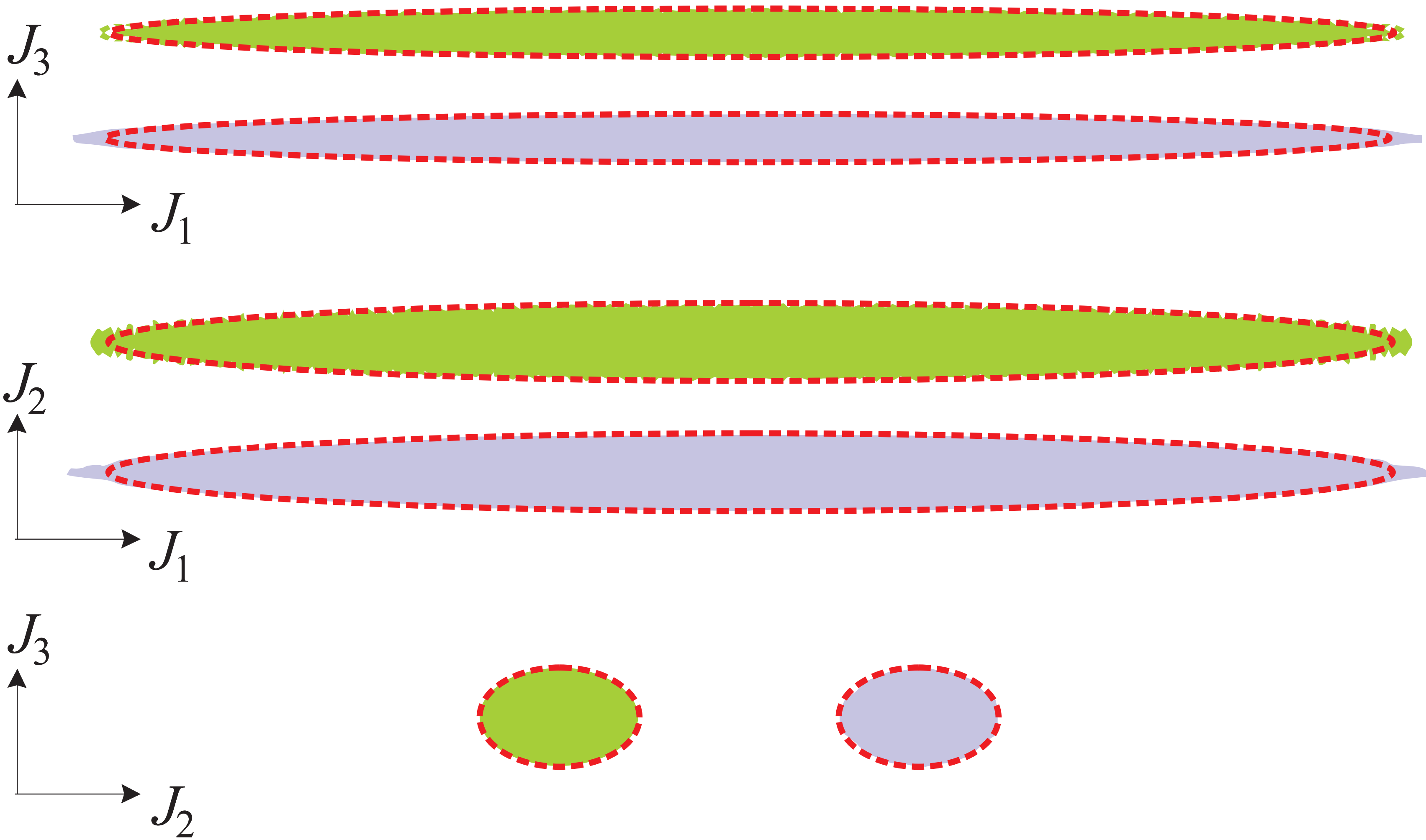
Ch Müller et al., New J. Phys.. **14**, 220401 (2012)

Reconstructed Wigner function

Isosurface of the reconstructed Wigner function in Poincaré space

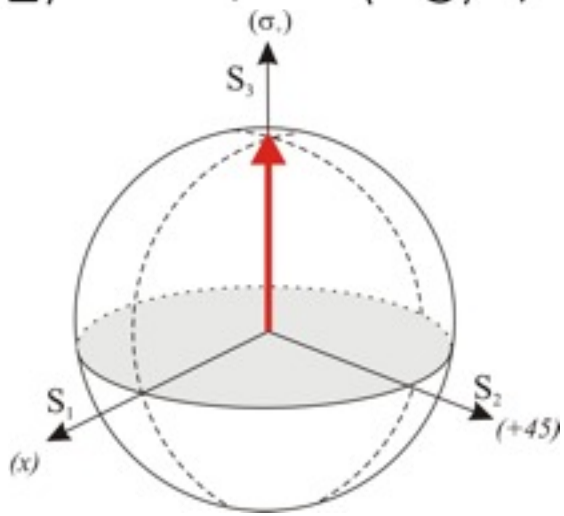


Reconstructed Wigner function



Special case: the dark plane

Right-handed circularly polarized light $\Rightarrow \langle S_1 \rangle = \langle S_2 \rangle = 0, \quad \langle S_3 \rangle \neq 0$



Relevant uncertainty relation

$$\Delta^2 S_1 \, \Delta^2 S_2 \geq |\langle S_3 \rangle|^2$$

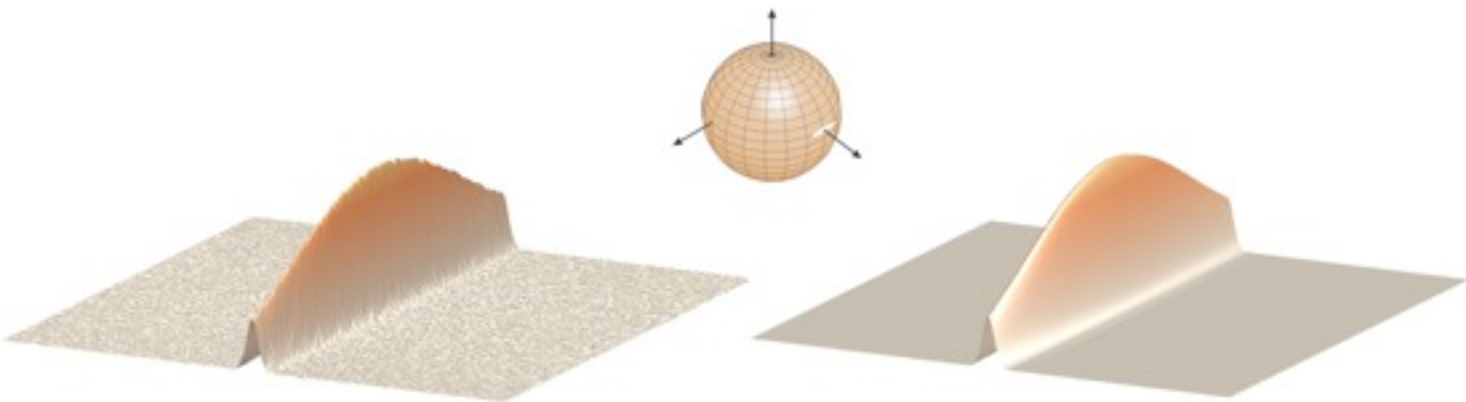
Stokes fluctuations

$$\delta S_\theta = \cos\theta \, \delta S_1 + \sin\theta \, \delta S_2 = \frac{\alpha}{\sqrt{2}} (\delta X_{H,\theta} - \delta X_{V,\theta})$$

$$\delta X_{H,\theta} = e^{i\theta} a_H^\dagger + e^{-i\theta} a_H$$

→ equivalent to (*x*, *p*) phase space

The dark plane



The "non-redundant" inversion

Idea: reconstruct moment by moment

First-order moments (Classical polarization)

$$J_{\mathbf{n}}(\theta,\phi) = f_{11}^{(J)} \sqrt{\frac{4\pi}{2J+1}} \sum_{q=-1}^{1} \varrho_{1q}^{(J)} Y_{1q}(\theta,\phi)$$

It is enough to measure in 3 directions and solve the system

$$\begin{pmatrix} \varrho_{11}^{(J)} \\ \varrho_{10}^{(J)} \\ \varrho_{1-1}^{(J)} \end{pmatrix} = \sqrt{\frac{3}{2J(J+1)(2J+1)}} \begin{pmatrix} 0 & -1 & i \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & i \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}$$

The "non-redundant" inversion

Second-order moments

$$J_{\mathbf{n}}^2(\theta,\phi) = \frac{1}{2J+1} f_{02}^{(J)} + f_{S22}^{(J)} \sqrt{\frac{4\pi}{2J+1}} \sum_{q=-2}^{2} \varrho_{2q}^{(J)} Y_{2M}(\theta,\phi)$$

It is enough to measure in 5 directions and solve the system

$$\mathbf{n}_{1,2} = \begin{pmatrix} 0 \\ \pm 2 \\ 1+\sqrt{5} \end{pmatrix} \qquad \mathbf{n}_{3,4} = \begin{pmatrix} \pm 2 \\ 1+\sqrt{5} \\ 0 \end{pmatrix} \qquad \mathbf{n}_{5} = \begin{pmatrix} +\sqrt{5} \\ 0 \\ 2 \end{pmatrix}$$

Conclusions

- Quantum optics entails polarization states that cannot be suitably described by the classical formalism.

- A complete tomography of polarization states is possible, but extremely demanding for it involves all the moments of the Stokes variables.

- A much more economic and feasible tomography is possible if we explore the state moment by moment.