A handy toolbox for picturing qubits in phase space

qubism

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Outline

- **Motivation**
- **Phase space for continuous variables**
- **Phase space for a single qudit**
- **Phase space for multiqudit systems**
- **Conclusions**

Motivation

- **Phase-space methods**
  - ✓ Quantum mechanics appears as a statistical theory in phase space
  - ✓ Simple to understand (no abstract Hilbert-space concepts)
  - ✓ Simple to picture
  - ✓ Computationally efficient

- **Basic ingredients**
  - ✓ (Classical) phase space
  - ✓ (Quasi) distributions in phase space
  - ✓ Coherent states
  - ✓ Star product

- **Drawbacks**
  - ✓ The machinery works well for continuous variables (and symmetry)
  - ✓ For discrete systems we need a toolbox to deal with them

Dynamical symmetry group

- ✓ Algebra of observables: self-adjoint position and momentum operators

$$[\hat{q}, \hat{p}] = i$$

Heisenberg-Weyl

- ✓ Phase space: coadjoint orbit associated with an irreducible representation of the dynamical symmetry group: $\mathbb{R}^2$

- ✓ Generators of translations in position and momentum

$$\hat{U}(q) = \exp(-iq\hat{p}) \qquad \hat{V}(p) = \exp(ip\hat{q})$$

$$\hat{U}(q')|q\rangle = |q + q'\rangle \qquad \hat{V}(p')|p\rangle = |p + p'\rangle$$

Phase space for continuous variables (II)

- ✓ Weyl unitary form

$$\hat{V}(p)\hat{U}(q) = e^{iqp}\hat{U}(p)\hat{V}(p)$$

- ✓ Displacement operators

$$\hat{D}(q,p) = e^{-iqp/2}\,\hat{U}(p)\hat{V}(q)\,, = \exp[i(p\hat{q} - q\hat{p})]$$

- ✓ Complete orthonormal set in the space of operators acting on $\mathcal{H}$

$$\mathrm{Tr}[\hat{D}(q,p)\,\hat{D}^{\dagger}(q',p')] = 2\pi\,\delta(q-q')\delta(p-p')\,.$$

Phase space for continuous variables (III)

- Coherent states

$$|q,p\rangle = \hat{D}(q,p)\,|\psi_0\rangle$$

- Fiducial state $|\psi_0\rangle$
  - Fundamental state of the harmonic oscillator (Gaussian!!)
  - Minimum uncertainty state
  - Eigenstate of the Fourier transform

Wigner function

- ✓ Map the density matrix into a classical function on phase space

$$W(q,p) = \mathrm{Tr}[\hat{\varrho}\,\hat{w}(q,p)]$$

$$\hat{\varrho} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{w}(q,p) W(q,p)\, dq dp$$

- ✓ Wigner kernel (Stratonovich-Weyl quantizer)

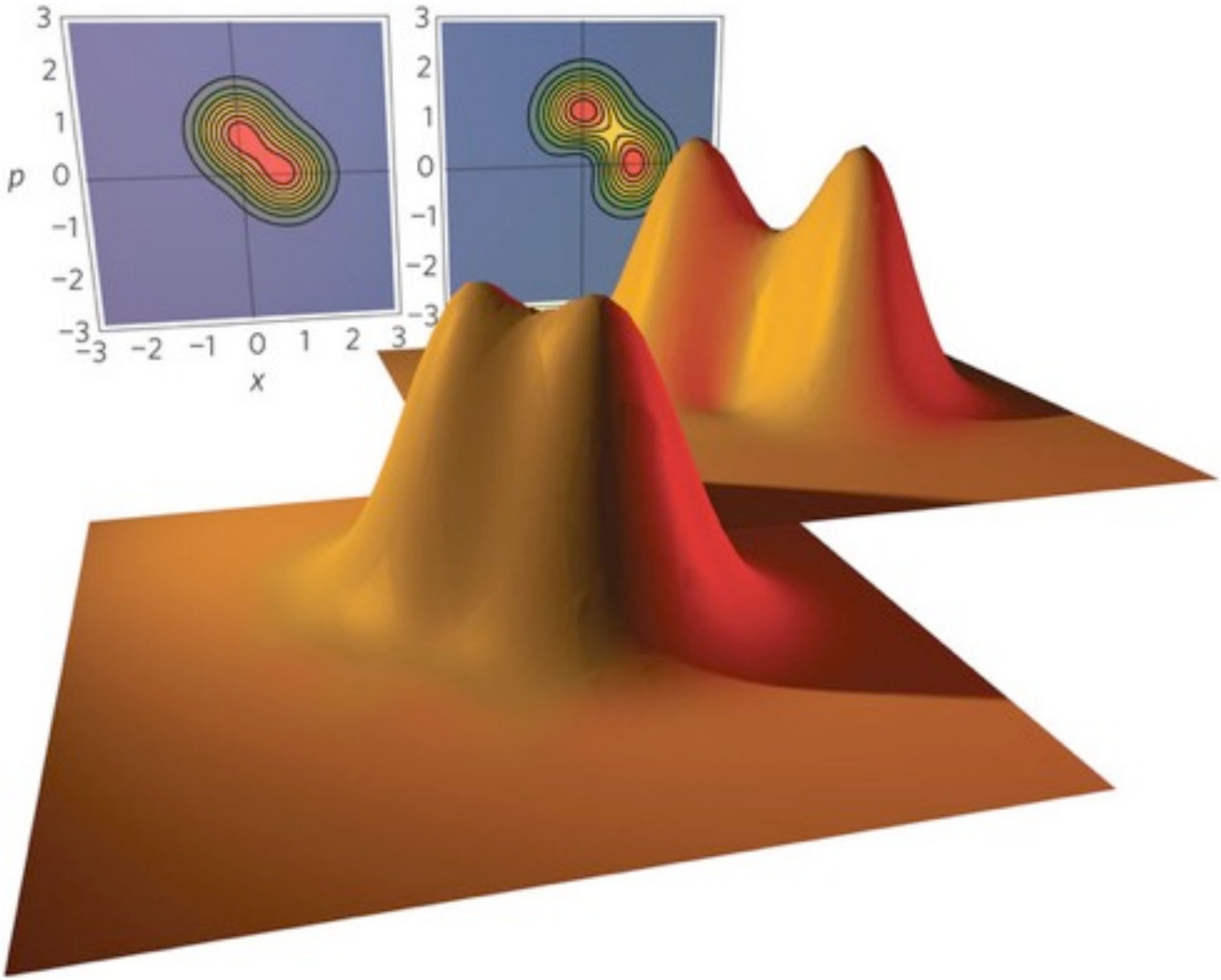
$$\hat{w}(q,p) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \exp[-i(pq' - qp')]\, \hat{D}(q',p')\, dq' dp'$$

Double Fourier transform

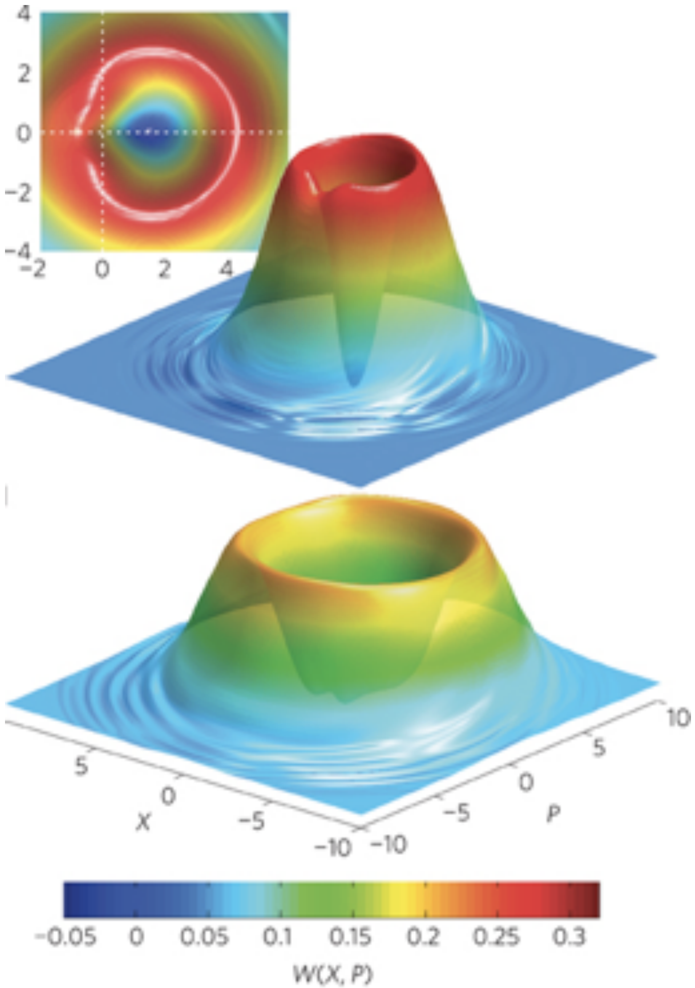
$$\hat{w}(q,p) = \hat{D}(q,p)\,\hat{w}(0,0)\,\hat{D}^\dagger(q,p) = 2\hat{D}(q,p)\,\hat{P}(0,0)\,\hat{D}^\dagger(q,p)$$

Displaced Parity

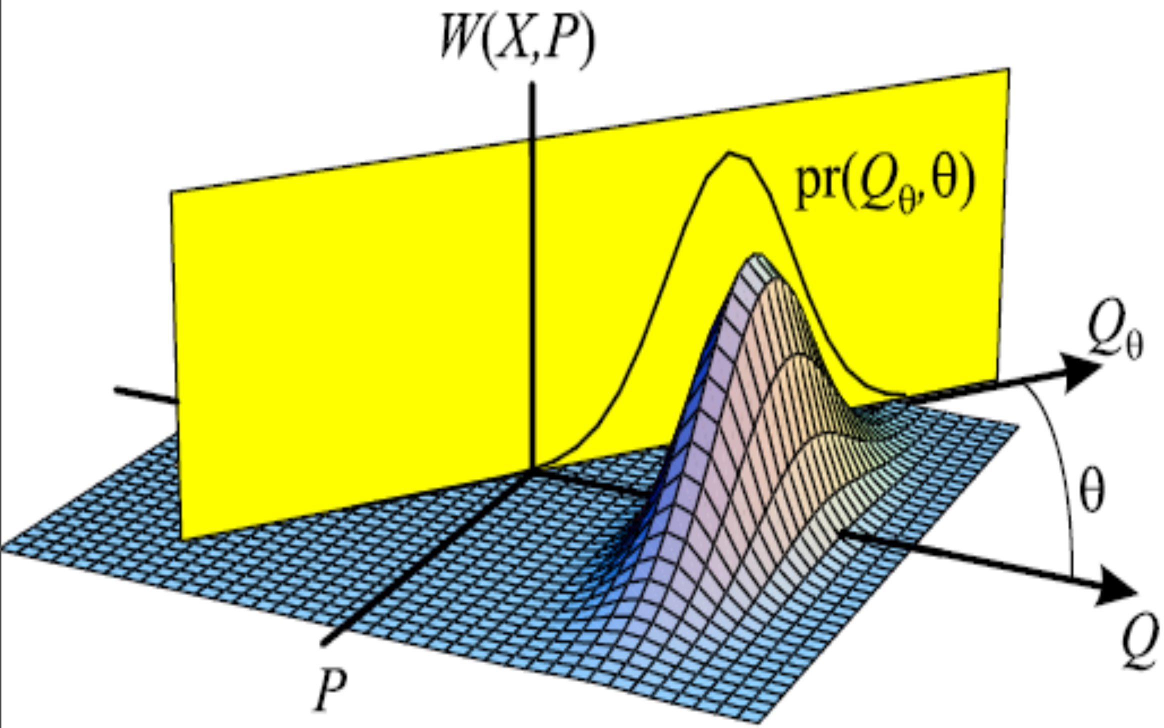
Phase space for continuous variables (IV)



Phase space for continuous variables (IV)



Phase space for continuous variables (V)



Properties

- ✓ Real
- ✓ Proper marginals
- ✓ Covariance

$$W_{\hat{\varrho}'}(q,p) = W_{\hat{\varrho}}(q-q_0, p-p_0)$$

- ✓ Traciality

$$\mathrm{Tr}(\hat{\varrho}_1\, \hat{\varrho}_2) \propto \int_{\mathbb{R}^2} W_1(q,p) W_2(q,p)\, dq dp$$

Phase space for continuous variables (VI)

- ✓ Symbol of an operator

$$\hat{A} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} a(q,p)\, \hat{w}(q,p)\, dq dp$$

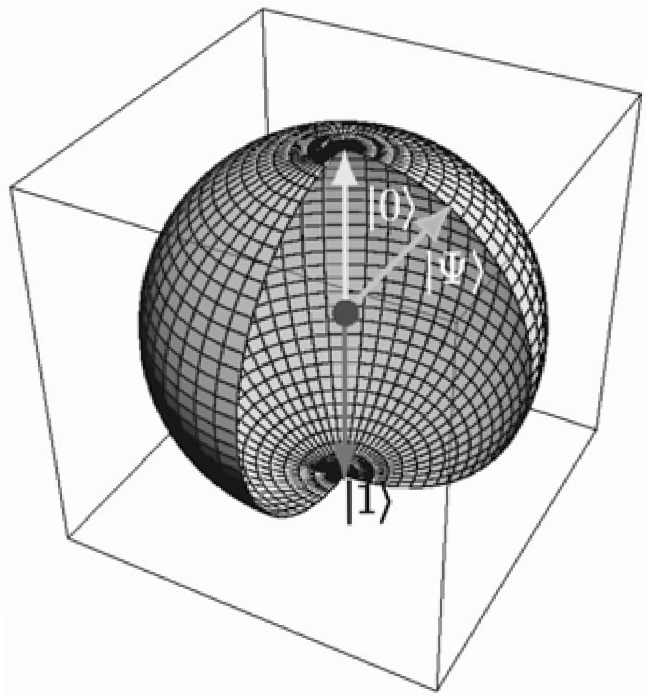
- ✓ Star product

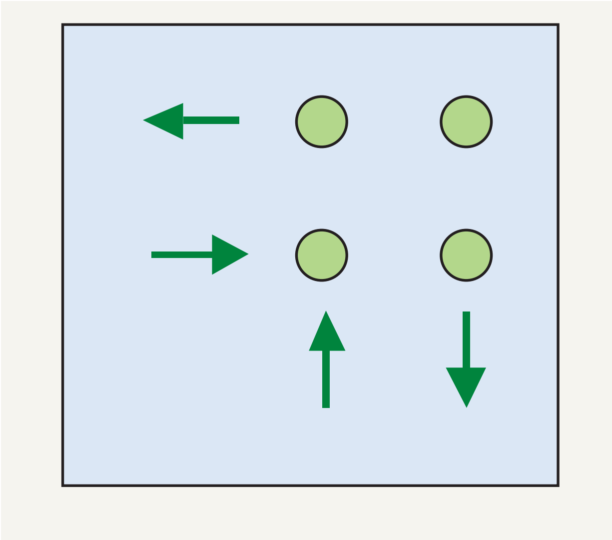
$$(a \star b)(q,p) = \frac{4}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} a(q+q', p+p')\, \exp[2i(q'p'' - q''p')]\, b(q+q'', p+p'')\, dq'dp'dq''dp''$$

$$(a \star b)(q,p) = a(q,p)\, \exp\left(-\frac{i}{2} \overleftrightarrow{\mathcal{P}}\right)\, b(q,p)$$

$$\overleftrightarrow{\mathcal{P}} = \frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q}$$ Poisson operator

Phase space for a single qudit (I)





A lot of redundant information!

The precise amount of information!

The phase space is a $d \times d$ grid of discrete points!

Phase space for a single qudit (II)

Building the geometry (d is a prime number)

- ✓ Computational and conjugate basis

"position" $|\ell\rangle$

$|\tilde{\ell}\rangle = \mathcal{F}\,|\ell\rangle$ "momentum"

$$\hat{\mathcal{F}} = \frac{1}{\sqrt{d}} \sum_{\ell,\ell'=0}^{d-1} \omega(\ell\,\ell')\,|\ell\rangle\langle\ell'|$$

- ✓ Generators of translations in position and momentum

$$\hat{U}^n|\ell\rangle = |\ell + n\rangle$$

$$\hat{V}^m|\ell\rangle = \omega(m\ell)\,|\ell\rangle$$

All the operations mod d

$$\omega(\ell) \equiv \omega^{\ell} = \exp(i2\pi\ell/d)$$

Phase space for a single qudit (III)

Remarks

- ✓ "Complementarity"

$$\hat{V} = \hat{\mathcal{F}}\,\hat{U}\,\hat{\mathcal{F}}^{\dagger}$$

- ✓ Weyl commutation relation

$$\hat{V}^{m}\hat{U}^{n} = \omega(mn)\,\hat{U}^{n}\hat{V}^{m}$$

- ✓ Discrete "position" and "momentum" operators?

$$\hat{U} = \exp(-2\pi i\hat{P}/d) \qquad \hat{V} = \exp(2\pi i\hat{Q}/d)$$

There are no infinitesimal displacements!

Phase space for a single qudit (IV)

- Displacement operators

$$\hat{D}(m,n) = e^{i\phi(m,n)}\,\hat{U}^n\hat{V}^m$$

Relevant choice $\phi(m,n) = \frac{2\pi}{d}2^{-1}\,mn$

- Coherent states

$$|m,n\rangle = \hat{D}(m,n)\,|\psi_0\rangle$$

- Properties analogous to their continuous counterparts.

Phase space for a single qudit (V)

Fixing the fiducial state

✓ Ground state of the "discrete harmonic oscillator" with periodic boundary conditions (Harper Hamiltonian)

$$\hat{H} = 2 - \frac{\hat{U} + \hat{U}^{\dagger}}{2} - \frac{\hat{V} + \hat{V}^{\dagger}}{2}$$

✓ Minimum uncertainty state

✓ Eigenstate of the discrete Fourier transform

$$|\psi_0\rangle = \frac{1}{\sqrt{C}} \sum_{\ell} \vartheta_3 \left(\frac{\pi \ell}{d} \Big| e^{-\frac{\pi}{d}} \right) |\ell\rangle$$

Phase space for a single qudit (VI)

Wigner function

- ✓ Map the density matrix into a classical function on the discrete grid

$$W_{\hat\varrho}(m,n) = \mathrm{Tr}[\hat\varrho\,\hat\Delta(m,n)]$$

$$\hat\varrho = \frac{1}{d}\sum_{m,n} W(m,n)\hat\Delta(m,n)$$

- ✓ Discrete Wigner kernel

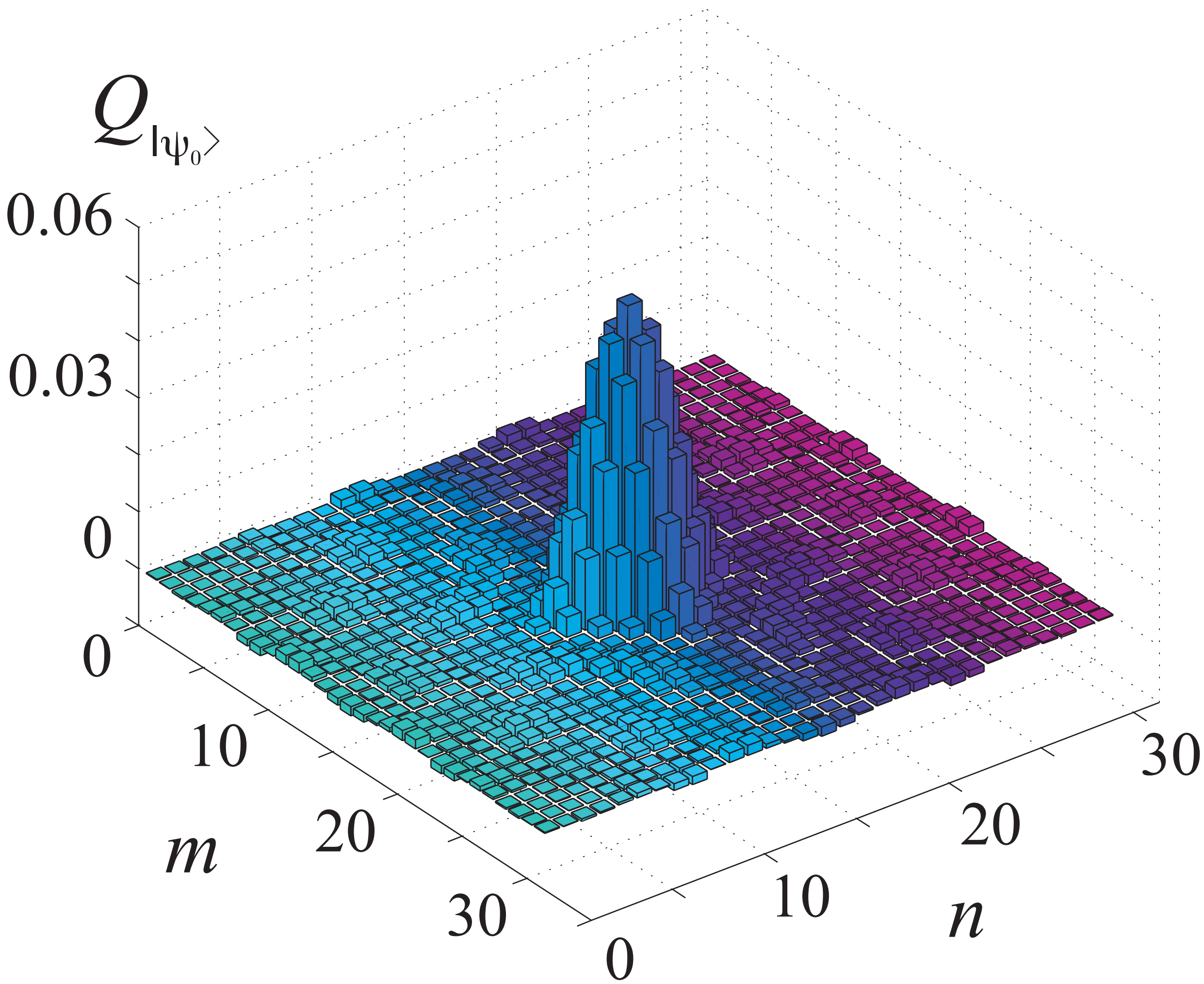
$$\hat\Delta(m,n) = \frac{1}{d}\sum_{k,l}\omega(nk-ml)\,\hat D(m,n)$$

Double Fourier transform

$$\hat\Delta(m,n) = \hat D(m,n)\,\hat\Delta(0,0)\,\hat D^\dagger(m,n) = 2\hat D(m,n)\,\hat P\,\hat D^\dagger(m,n)$$

Displaced Parity

Phase space for a single qudit (VII)



Phase space for multiqudit systems (I)

✓ Instead of natural numbers, it is then convenient to use elements of the finite field Gal(d^n) to label states.

✓ We can almost directly translate all the properties for a single qudit and we can endow the phase-space with the geometrical properties of the ordinary plane.

✓ Let $|\lambda\rangle$ be a computational basis (labeled by powers of a primitive element in the field)

$$\hat{U}_\nu|\lambda\rangle = |\lambda + \nu\rangle$$

$$\hat{V}_\mu|\lambda\rangle = \chi(\mu\lambda)|\lambda\rangle$$

Phase space for multiqudit systems (II)

- ✓ Displacement operators

$$\hat{D}(\mu,\nu) = \phi(\mu,\nu)\,\hat{U}_\nu \hat{V}_\mu \qquad \phi(\mu,\nu) = \chi(2^{-1}\mu\nu)$$

- ✓ Coherent states

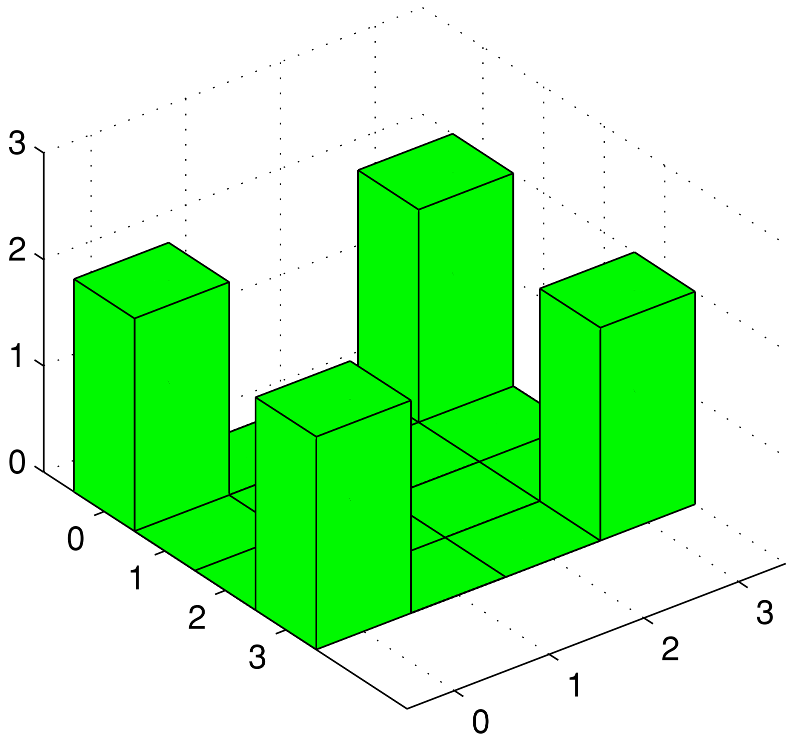
$$|\mu,\nu\rangle = \hat{D}(\mu,\nu)|\Psi_0\rangle$$

- ✓ Wigner function

$$W_{\hat{\varrho}}(\mu,\nu) = \mathrm{Tr}[\hat{\varrho}\,\Delta(\mu,\nu)]$$

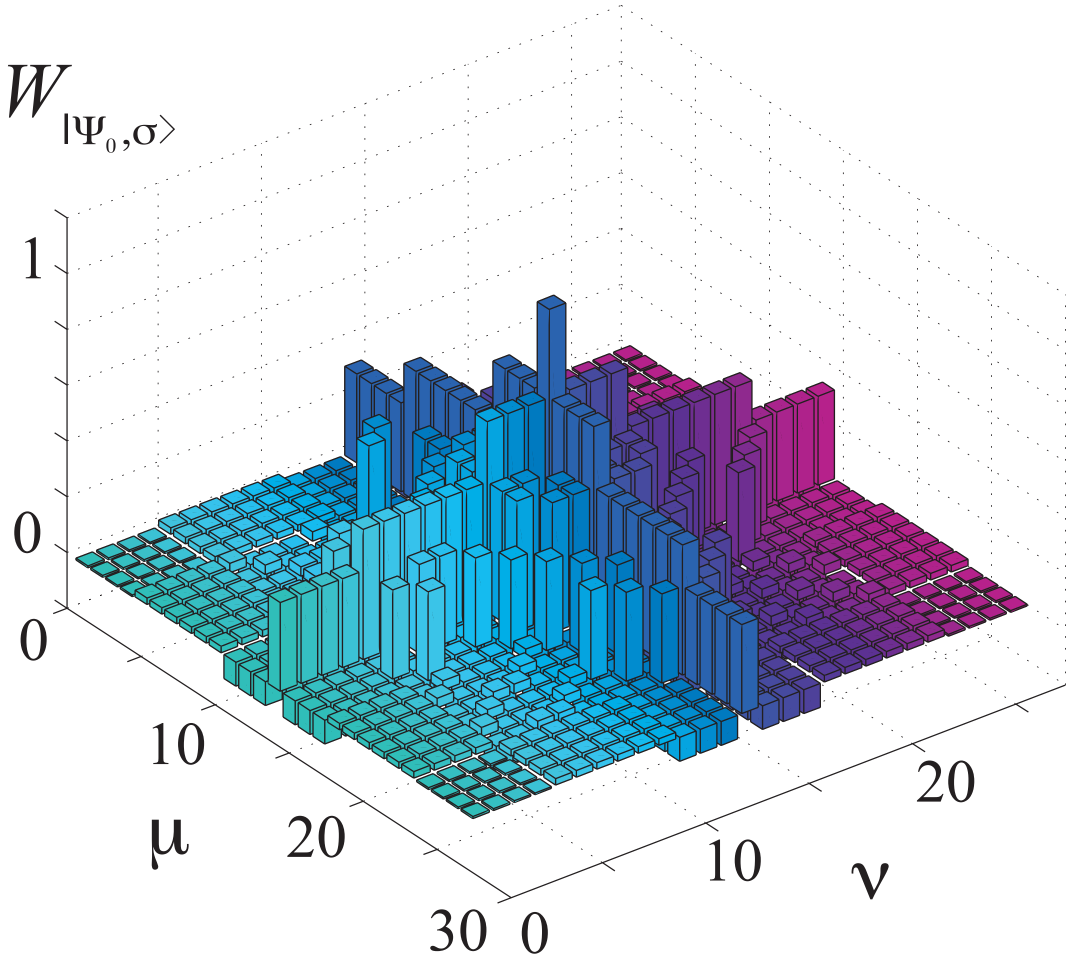
$$\Delta(\mu,\nu) = \frac{1}{d^n}\sum_{\lambda,\kappa}\chi(\mu\lambda - \nu\kappa)\,\hat{D}(\lambda,\kappa)$$

Phase space for multiqudit systems (III)



$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Phase space for multiqudit systems (IV)



Conclusions

- For many purposes is useful to have a way of picturing quantum states. We have provided a comprehensive toolbox for this task.

- All the relevant quantum mechanical aspects can be encompased within this framework, provided one uses the proper phase space.

- The role of the Fourier transform in quantum information can be never underestimated.