Cavity QED and Trapped Ions

E. Solano

UPV/EHU and Ikerbasque, Bilbao

Theory and exercises

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Chapter II: Atoms, EM fields, and motion

II.a) Trapped ion physics

II.b) Red and blue sideband excitations

II.c) Measurement of a qubit: electron shelving

II.d) Measurement of the Wigner function

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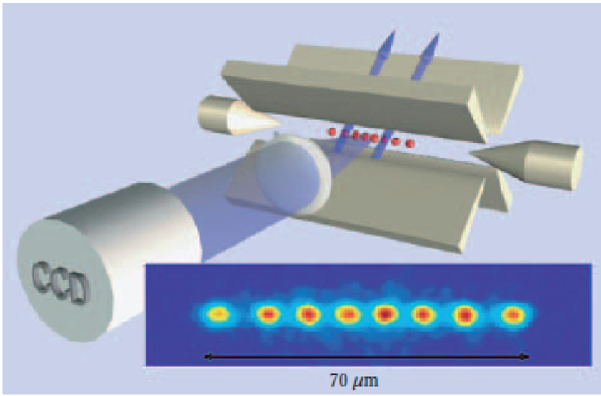
II.a) Trapped ion physics

Charged atoms (ions) can be trapped in a combination of axial electrostatic and transversal oscillating quadrupolar potentials. The most used is the linear **Paul trap** where the radial confinement can be very strong compared to the axial one, allowing a unidimensional treatment of the motion of the trapped ions.



Innsbruck Ion Trap

In addition, lasers acting on the ions can couple their internal degrees of freedom with their motion, which can be cooled down to very low temperatures, where a **quantized treatment of the motion** is required. In this case we could write the following Hamiltonian

$$H = H_o + H_{\text{int}}$$

where

$$H_o = \frac{\hbar\omega_o}{2}\sigma_z + \frac{p^2}{2m} + \frac{m\nu^2 z^2}{2m} = \frac{\hbar\omega_o}{2}\sigma_z + \hbar\nu(a^\dagger a + \frac{1}{2})$$

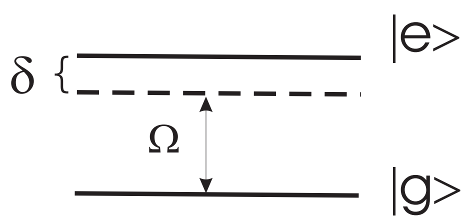
and

$$H_{\text{int}} = \hbar\Omega(\sigma^+ + \sigma^-)[e^{i(qz-\omega t+\phi)} + e^{-i(qz-\omega t+\phi)}].$$

Here, the position and momentum operators are, respectively,

$$z = \sqrt{\frac{\hbar}{2m\nu}}(a^\dagger + a),$$

$$p = i\sqrt{\frac{m\hbar\nu}{2}}(a^\dagger - a).$$



In the interaction picture, and after the optical RWA approximation,

$$H^I_{\text{int}} = \hbar\Omega\{\sigma^+ \exp[i\eta(a^\dagger e^{i\nu t} + a e^{-i\nu t}) - i(\delta t - \phi)] + \text{H.c.}\}.$$

Here, $\delta = \omega - \omega_o$ and

$$\eta = q\sqrt{\frac{\hbar}{2m\nu}} = q\Delta z = 2\pi\frac{\Delta z}{\lambda} = 2\pi\frac{\sqrt{\langle 0|z^2|0\rangle}}{\lambda}$$

is the **Lamb-Dicke parameter**, proportional to the ratio between the width of the ground-state wave function and the excitation wavelength λ.

II.b) Red and blue sideband in trapped ions

By choosing $\delta = -k\nu$ (k positive integer), replacing it in H^I_{int}, and neglecting fast oscillating terms, also called **vibrational RWA**, we have

$$H^I_R = \hbar\Omega[\sigma^+ F_k(a^\dagger a) a^k e^{i\phi} + \sigma^- a^{\dagger k} F^\dagger_k(a^\dagger a) e^{-i\phi}] \,.$$

with

$$F_k(a^\dagger a) = e^{-\eta^2/2} \sum_{\ell=0}^{\infty} \frac{(i\eta)^{2\ell+k} a^{\dagger\ell} a^\ell}{\ell!(\ell+k)!} \,.$$

Note that F_k is a polynomial in the number operator and does not change the motional populations. The Hamiltonian H_R represents a **nonlinear JC interaction for the k-th red motional sideband**.

The temporal evolution of the initial state $|e\rangle|n\rangle$ is

$$|\Psi(t)\rangle = \cos(\Omega^k_n t)|e\rangle|n\rangle - ie^{-i\phi}\sin(\Omega^k_n t)|g\rangle|n+k\rangle$$

where

$$\Omega^k_n = (i\eta)^k e^{-\eta^2/2}\Omega\sqrt{\frac{n!}{(n+k)!}} L^k_n(\eta^2)$$

are the Rabi frequencies associated with the transitions $n \leftrightarrow n+k$, and L^k_n are generalized Laguerre polynomial.

By choosing $\delta = k\nu$ (k positive integer), replacing it in $H^I_{\rm int}$, and neglecting fast oscillating terms, we have

$$H^I_B = \hbar\Omega[\sigma^+ a^{\dagger k} F_k(a^\dagger a) e^{i\phi} + \sigma^- F^\dagger_k(a^\dagger a) a^k e^{-i\phi}].$$

The Hamiltonian H_B represents a **nonlinear anti-JC interaction for the k-th blue motional sideband**.

A third case of interest happens when $k = 0$. This yields, in the Lamb-Dicke regime, a **nonlinear carrier excitation**

$$H^I_{\rm C} = \hbar\Omega(\sigma^+ F_o(a^\dagger a) e^{i\phi} + \sigma^- F_o(a^\dagger a) e^{-i\phi}),$$

which produces nonlinear rotations depending on the number of phonon excitations.

Lamb-Dicke regime: JC and anti-JC in ions

In particular, the so-called Lamb-Dicke regime emerges when $\eta\sqrt{\overline{n}} \ll 1$ ($L^k_n \to 1$).

If in addition we choose $k = 1$, we can write

$$H^I_{\rm JC} = i\hbar\eta\Omega(\sigma^+ a e^{i\phi} - \sigma^- a^\dagger e^{-i\phi}),$$

out of H_R, which represents the implementation of the **JC model in trapped ions**.

If we choose $k = -1$, then we can write

$$H^I_{\mathsf{AJC}} = i\hbar\eta\Omega(\sigma^+ a^\dagger e^{i\phi} - \sigma^- a e^{-i\phi}),$$

out of H_B, which represents the implementation of the **anti-JC (AJC) model in trapped ions**.

A third case of interest happens when $k = 0$. This yields, in the Lamb-Dicke regime, a **carrier excitation**

$$H^I_{\mathsf{C}} = \hbar\Omega(\sigma^+ e^{i\phi} + \sigma^- e^{-i\phi}),$$

which produces rotations around x-axis ($\phi = 0$) and y-axis ($\phi = \pi/2$) in a trapped ion.

Exercise II.1: Deduce ab initio the effective Hamiltonians H^I_{JC}, H^I_{AJC}, and H^I_{C}, indicating the approximations involved.

Simultaneous JC + anti-JC in trapped ions

It is possible to consider a **bichromatic excitation** of blue and red sidebands of a trapped ion, yielding effective Hamiltonians of different kinds. For example, we could build the following cases

$$\begin{aligned} H^I_{\sigma_{\mathsf{x}}\mathsf{x}} &= \hbar\eta\Omega(\sigma^+ + \sigma^-)(a^\dagger + a) \propto \sigma_x . x \\ H^I_{\sigma_{\mathsf{x}}\mathsf{p}_{\mathsf{x}}} &= \hbar\eta\Omega(\sigma^+ + \sigma^-)(a^\dagger - a)/i \propto \sigma_x . p_x \\ H^I_{\sigma_{\mathsf{y}}\mathsf{x}} &= \hbar\eta\Omega(\sigma^+ - \sigma^-)(a + a^\dagger)/i = \sigma_y . x \\ &\cdots \\ &\cdots \end{aligned}$$

This kind of interactions generates Schrödinger cats in a resonant (fast) manner. We will also see later that suitable combinations of these Hamiltonians could implement **quantum simulations of the Dirac equation in 3+1, 2+1, and 1+1 dimensions**.

For more detail, see Phys. Rev. Lett. 87, 060402 (2001); Phys. Rev. Lett. 94, 153602 (2005); Phys. Rev. Lett. 98, 253005 (2007); http://arxiv.org/abs/0909.0674.

Exercise II.2: The Dirac equation in 1+1 dimensions, in a variant of the supersymmetric representation, can be written as

$$i\hbar\frac{d}{dt}|\Psi\rangle = H_D|\Psi\rangle$$

with

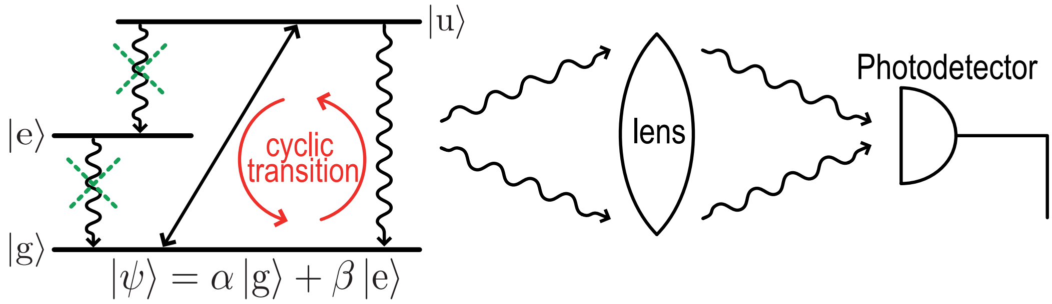
$$H_D = c\sigma_x p_x + mc^2\sigma_z.$$

a) By using bichromatic excitations, show that it is possible to implement the quantum simulation of the Dirac equation with 1+1 dimensions in a single trapped ion.

b) Show how to tune the speed of light and the mass of the Dirac particle in your proposed simulation.

II.c) Measurement of a qubit: electron shelving

Electron shelving is the most precise qubit-readout technique in any quantum mechanical system. It has reached **$99,99\%$ of fidelity** in the discrimination between the ground and the excited state of a qubit encoded in a trapped ion.



In each shelving cycle, a photon is emitted and the overall efficiency of detecting it is $\eta \ll 1$. After N cycles, the probability of sequential failure is $(1-\eta)^N = (1-\eta N/N)^N \sim \exp(-\eta N)$, and the final **fidelity** can be estimated as

$$F = 1 - e^{-\eta N}.$$

For more information, see Rev. Mod. Phys. 75, 281 (2003); Phys. Rev. Lett. 100, 200502 (2008).

II.d) Measurement of the Wigner function

Method of the Fresnel transform

Exercise II.3:

a) Consider the initially decoupled atom-field state (be phonon or photon field)

$$|\Psi(0)\rangle = |e\rangle \otimes \sum_n c_n|n\rangle,$$

where the **field state** $\sum_n c_n|n\rangle$ **is unknown to us**, *and show that, after an interaction time* τ *of a red sideband (JC) excitation, it is possible to estimate the population of the ground state as*

$$P_g(\tau) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} P_n \cos(2g\sqrt{n+1}\tau),$$

where $P_n = |c_n|^2$ *is the unkown population of the initial field.*

b) Let us displace previously the initial field state with the displacement operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$,

$$|\Psi_\alpha(0)\rangle = |e\rangle \otimes D(\alpha)\sum_n c_n|n\rangle = |e\rangle \otimes \sum_n c_n(\alpha)|n\rangle,$$

and show that, after an interaction time τ *of a JC evolution,*

$$P_g(\tau;\alpha) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} P_n(\alpha)\cos(2g\sqrt{n+1}\tau),$$

where $P_n(\alpha) = |c_n(\alpha)|^2$ *is the population of the displaced field.*

c) Assuming we are able to measure in the lab $P_g(\tau;\alpha)$ *for different complex displacements "*α*" and interaction times "*τ*", suggest a* **method for recovering the information** *about* $P_n(\alpha) = |c_n(\alpha)|^2$.

d) Are we able to **retrieve the full complex amplitudes** $c_n's$ *from the previous knowledge? Note that this would amount to a* **full state reconstruction** *of the initially unknown field state.*

e) From **pseudoprobability distributions** *in phase space, the* **Wigner function** $\mathrm{W}(\alpha)$, *with* $d^2\alpha = d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha) \propto dx\,dp$, *is the most reputed. The reason is that the Wigner function contains the same information as the field density matrix and, furthermore, its* **3D plots display better the interference and nonclassical features** *of certain field states.*

Following Glauber, the Wigner function can be defined as the Fourier transform of the characteristic function $\chi(\xi) = \mathrm{Tr}[\rho D(\xi)]$,

$$W(\alpha) = \frac{1}{\pi}\int e^{\alpha\xi^* - \alpha^*\xi}\,\chi(\xi)\,d^2\xi\,,$$

We can retrieve the density operator of the system by inverting these operations,

$$\rho = \frac{1}{\pi}\int W(\alpha)T(\alpha)\,d^2\alpha,$$

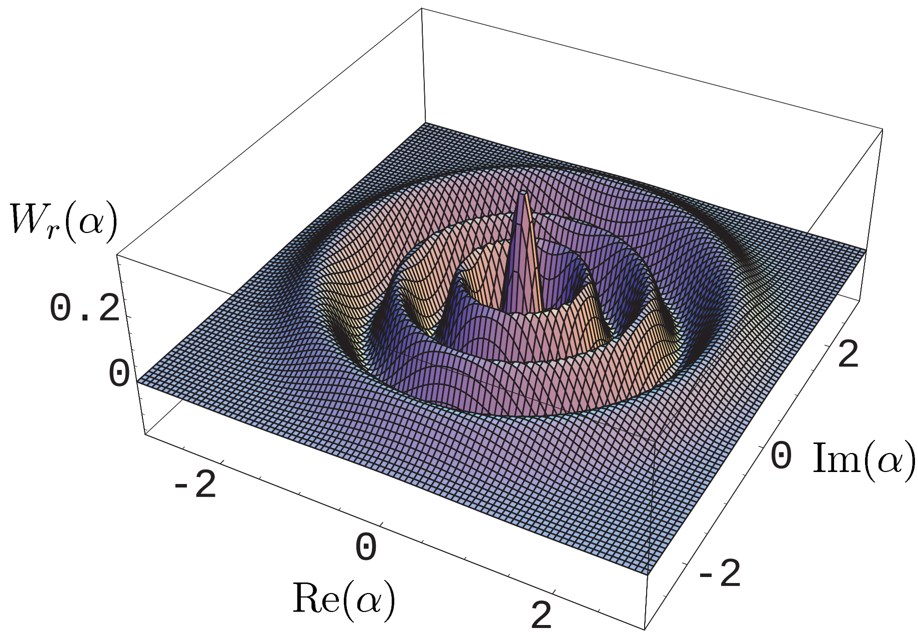
where the operator $T(\alpha) = \frac{1}{\pi}\int D(\xi)e^{\alpha\xi^* - \alpha^*\xi}\,d^2\xi$.

A much simpler expression to calculate the Wigner function is

$$W(\alpha) \equiv 2\sum_{n=0}^{\infty}(-1)^n P_n(-\alpha),$$

where $P_n(-\alpha) = |c_n(-\alpha)|^2$ *is the population of photon/phonon field displaced in the complex amplitude* $(-\alpha)$.

Calculate and make a 3D plot of the Wigner function associated with field states $|0\rangle$, $|1\rangle$, $(|0\rangle + |1\rangle)/\sqrt{2}$, $|\alpha\rangle$, and $\mathcal{N}_{\pm}(|\alpha\rangle \pm |-\alpha\rangle)$.



Wigner function of a Fock state with six excitations: $|6\rangle$.

f) Show that the Fresnel integral of cosine satisfies

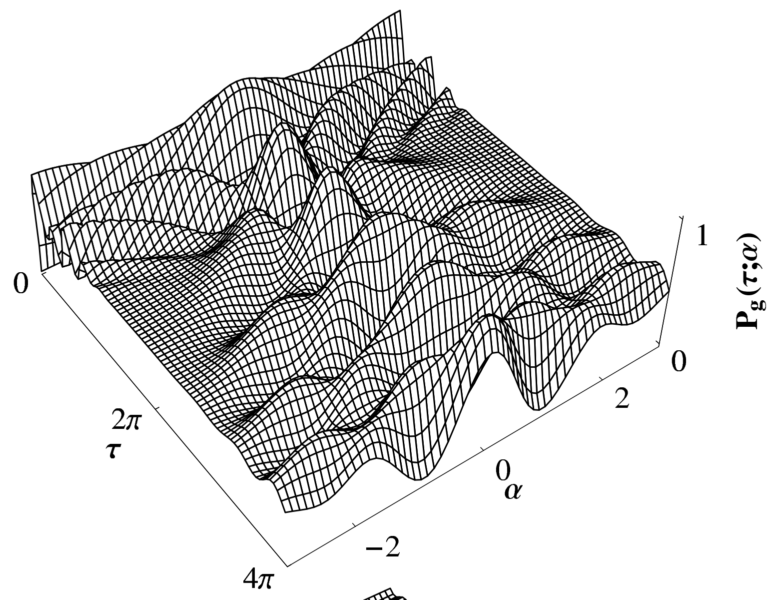
$$\frac{2}{\pi\sqrt{i}}\int_0^\infty d\tau \exp\left(i\tau^2/\pi\right)\cos(2\sqrt{n}\tau) = (-1)^n,$$

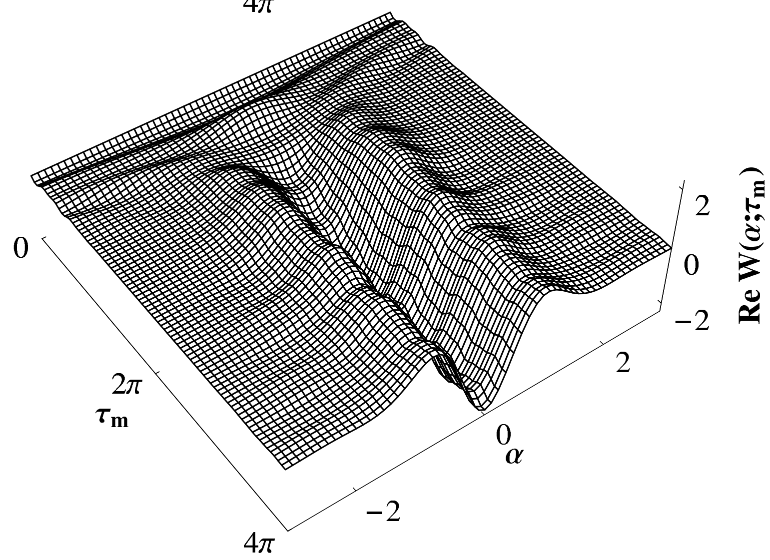
and that, finally, it is possible to **reconstruct the Wigner function from measured $P_g(\tau;-\alpha)$ via the Fresnel integral**

$$W(\alpha) \equiv 2\sum_{n=0}^{\infty}(-1)^n P_n(-\alpha) = \frac{8}{\pi\sqrt{i}}\int_0^\infty d\tau e^{i\tau^2/\pi}\left[P_g(\tau;-\alpha)-\frac{1}{2}\right].$$

In conclusion, we can obtain the Wigner function directly from the atomic dynamics $P_g(\tau;-\alpha)$ without the necessity to know each P_n.

For a review, see the masterpiece K. E. Cahill and R. J. Glauber, Phys. Rev. 177, 1882 (1969); and for the Fresnel transform method see Phys. Rev. Lett. 91, 010401 (2003).





Wigner function reconstruction via the Fresnel transform method. The upper figure displays Rabi oscillations of displaced initial Fock state $|1\rangle$. The lower figure represents the reconstruction of the Wigner funtion through Fresnel kernel integration of $P_g(\tau;-\alpha)$ until truncated time τ_m.

II.e) Instantaneous measurements of the motion

Many times we do not want or need the whole density matrix of the system, but just **a specific expectation value of a signficant observable, say a field quadrature or the mean number of photons**. In those cases, more efficient methods can be derived.

We will show how to **measure relevant field observables with the derivatives of the qubit population at vanishing interaction time,** $\tau = 0$.

We consider an initially decoupled qubit-field system described by state $\rho(0) = \rho_{\text{at}} \otimes \rho_f$. We can choose the initial qubit state ρ_{at} but we do not know ρ_f. After an interaction time t, **the probability of finding the ion in the excited state** $|e\rangle$ is given by

$$P_{\text{e}}(t) = \text{Tr}[\rho(t)|\text{e}\rangle\langle\text{e}|] = \langle|\text{e}\rangle\langle\text{e}|\rangle,$$

where $\rho(t)$ is the density operator of the coupled system after an interaction time t.

We know that for any operator A,

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle[A,H]\rangle + \langle\frac{dA}{dt}\rangle.$$

Then, for the time-independent operator $A = |\text{e}\rangle\langle\text{e}|$,

$$\frac{d}{dt}P_{\text{e}}(t) = \frac{1}{i\hbar}\langle[|\text{e}\rangle\langle\text{e}|, H]\rangle.$$

We consider now that the **qubit-field coupling follows the JC dynamics**, $H_{\mathsf{JC}} = H_o + H_{\mathsf{int}}$. In this case, $[|\mathsf{e}\rangle\langle\mathsf{e}|, H_o] = 0$, and

$$\frac{d}{dt}P_{\mathsf{e}}(t) = \frac{1}{i\hbar}\langle[|\mathsf{e}\rangle\langle\mathsf{e}|, H_{\mathsf{int}}]\rangle.$$

With the use of $H_{\mathsf{int}} = \hbar g(\sigma^+ a + \sigma^- a^\dagger)$, we can calculate $[|\mathsf{e}\rangle\langle\mathsf{e}|, H_{\mathsf{int}}] = \hbar g(\sigma^+ a - \sigma^- a^\dagger)$, obtaining

$$\frac{d}{d\tau}P_{\mathsf{e}}(\tau) = \frac{1}{i}\mathrm{Tr}[\rho(\tau)(\sigma^+ a - \sigma^- a^\dagger)],$$

where we have used the dimensionless time $\tau = gt$.

We consider now that the qubit is prepared initially in the state

$$\rho(0) = |+_\phi\rangle\langle+_\phi| \otimes \rho_{\mathsf{f}},$$

where $|+_\phi\rangle = \frac{1}{\sqrt{2}}(|g\rangle + e^{i\phi}|e\rangle)$ and ρ_{f} is the field state we aim at characterizing. Then, at interaction time $\tau = 0$,

$$\frac{d}{d\tau}P_{\mathsf{e}}^{+_\phi}(\tau)\bigg|_{\tau=0} = \frac{1}{i}\mathrm{Tr}[|+_\phi\rangle\langle+_\phi| \otimes \rho_f(\sigma^+ a - \sigma^- a^\dagger)],$$

yielding

$$\frac{d}{d\tau}P_{\mathsf{e}}^{+_\phi}(\tau)\bigg|_{\tau=0} = \langle X_{\phi+\frac{\pi}{2}}\rangle,$$

where the generalized field quadrature X_ϕ is defined as

$$X_\phi = \frac{ae^{-i\phi} + a^\dagger e^{i\phi}}{2}\ .$$

In particular, $X_{\phi=0}/X_{\phi=\pi/2}$ is proportional to the **position/momentum of a trapped ion or the electric/magnetic field in cavity QED**, respectively.

Exercise II.4:

a) Reproduce the previous results and, making use of these instantaneous measurements, explain how to measure the position/momentum operators of a single trapped ion, or the electric/magnetic fields of a cavity single mode.

b) Demonstrate that it is possible to measure the mean number of field excitations (photons/phonons) through the following expression

$$\langle n \rangle = \frac{1}{2}\frac{d^2 P^{\mathrm{e}}_{\mathrm{g}}(\tau)}{d^2\tau}\Big|_{\tau=0} - 1 \ ,$$

where $P^{\mathrm{e}}_{\mathrm{g}}(\tau)$ is the measured population of the ground state $|\mathrm{g}\rangle$ at interaction time τ, assuming that the initial state is $|\mathrm{e}\rangle$.