Classical information exchange and quantum entanglement between two coupled harmonic oscillators

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A dilettante's thoughts around entanglement, decoherence, and information dynamics

¿ How are correlations, entanglement (inseparability) and shared information related to one another?

¿ Can we pinpoint examples where correlations and entanglement prove to be manifestly distinct concepts?

¿ Is it possible to attribute the “sudden death of entanglement” to sign changes of correlation functions?

¿ How is the time evolution of these quantities related to the information flow between subsystems of a multi-partite system?

¿ What characterizes the information dynamics in linear (harmonic) vs. nonlinear systems?

¿ How exactly is information distributed and “absorbed” in heat baths?

A dilettante's thoughts around entanglement, decoherence, and information dynamics

*8. THE CORRELATION COEFFICIENT

Let $\mathbf{X}$ and $\mathbf{Y}$ be any two random variables with means μ_x and μ_y and positive variances σ_x^2 and σ_y^2. We introduce the corresponding normalized variables $\mathbf{X}^*$ and $\mathbf{Y}^*$ defined by (4.6). Their covariance is called *the correlation coefficient of* $\mathbf{X}, \mathbf{Y}$ *and is denoted by* $\rho(\mathbf{X}, \mathbf{Y})$. Thus, using (5.4),

$$\rho(\mathbf{X}, \mathbf{Y}) = \mathrm{Cov}\,(\mathbf{X}^*, \mathbf{Y}^*) = \frac{\mathrm{Cov}\,(\mathbf{X}, \mathbf{Y})}{\sigma_x \sigma_y}. \tag{8.1}$$

Clearly this correlation coefficient is independent of the origins and units of measurements, that is, for any constants a_1, a_2, b_1, b_2, with $a_1 > 0$, $a_2 > 0$, we have $\rho(a_1\mathbf{X}+b_1, a_2\mathbf{Y}+b_2) = \rho(\mathbf{X}, \mathbf{Y})$.

The use of the correlation coefficient amounts to a fancy way of writing the covariance.[8] Unfortunately, the term correlation is suggestive of implications which are not inherent in it. We know from section 5 that $\rho(\mathbf{X}, \mathbf{Y}) = 0$ whenever $\mathbf{X}$ and $\mathbf{Y}$ are independent. It is important to realize that the converse is not true. In fact, *the correlation coefficient* $\rho(\mathbf{X}, \mathbf{Y})$ *can vanish even if* $\mathbf{Y}$ *is a function of* $\mathbf{X}$.

Examples. (*a*) Let $\mathbf{X}$ assume the values $\pm 1, \pm 2$ each with probability $\frac{1}{4}$. Let $\mathbf{Y} = \mathbf{X}^2$. The joint distribution is given by $p(-1, 1) = p(1, 1) = p(2, 4) = p(-2, 4) = \frac{1}{4}$. For reasons of symmetry $\rho(\mathbf{X}, \mathbf{Y}) = 0$ even though we have a direct functional dependence of $\mathbf{Y}$ on $\mathbf{X}$.

(*b*) Let $\mathbf{U}$ and $\mathbf{V}$ have the same distribution, and let $\mathbf{X} = \mathbf{U} + \mathbf{V}$, $\mathbf{Y} = \mathbf{U} - \mathbf{V}$. Then $\mathbf{E}(\mathbf{XY}) = \mathbf{E}(\mathbf{U}^2) - \mathbf{E}(\mathbf{V}^2) = 0$ and $\mathbf{E}(\mathbf{Y}) = 0$. Hence $\mathrm{Cov}\,(\mathbf{X}, \mathbf{Y}) = 0$ and therefore also $\rho(\mathbf{X}, \mathbf{Y}) = 0$. For example, $\mathbf{X}$ and $\mathbf{Y}$ may be the sum and difference of points on two dice. Then $\mathbf{X}$ and $\mathbf{Y}$ are either both odd or both even and therefore dependent. ▶

It follows that the correlation coefficient is by no means a general measure of dependence between $\mathbf{X}$ and $\mathbf{Y}$. However, $\rho(\mathbf{X}, \mathbf{Y})$ is connected with the *linear* dependence of $\mathbf{X}$ and $\mathbf{Y}$.

Theorem. *We have always* $|\rho(\mathbf{X}, \mathbf{Y})| \leq 1$; *furthermore,* $\rho(\mathbf{X}, \mathbf{Y}) = \pm 1$ *only if there exist constants* a *and* b *such that* $\mathbf{Y} = a\mathbf{X} + b$, *except, perhaps, for values of* $\mathbf{X}$ *with zero probability.*

Proof. Let $\mathbf{X}^*$ and $\mathbf{Y}^*$ be the normalized variables. Then

$$\mathrm{Var}\,(\mathbf{X}^* \pm \mathbf{Y}^*) = \mathrm{Var}\,(\mathbf{X}^*) \pm 2\,\mathrm{Cov}\,(\mathbf{X}^*, \mathbf{Y}^*) + \mathrm{Var}\,(\mathbf{Y}^*) = 2(1 \pm \rho(\mathbf{X}, \mathbf{Y})). \tag{8.2}$$

* This section treats a special topic and may be omitted at first reading.

[8] The physicist would define the correlation coefficient as "dimensionless covariance."

"A vanishing correlation coefficient between two subsystems is compatible with these subsystems being statistically dependent or even functions of one another."

from William Feller:
"An Introduction to Probability Theory and Its Applications", Vol. 1, 3rd Ed.,
Wiley Series in Probability and Mathematical Statistics
John Wiley and Sons (1950)

Classical and quantum information dynamics in terms of information flow in phase space

Starting point

- In classical canonical as well as in quantum unitary dynamics, information (Shannon / von Neumann) is a conserved quantity.
- Hence, information flow between subsystems is a well-defined concept.

Objectives

- Analyze classical correlations and quantum entanglement in terms of information exchange among subsystems (degrees of freedom)
- Employ classical probability densities vs. quantum phase-space representations (e.g., the Wigner function) to resolve information dynamics on the level of phase-space structures and subsectors of Hilbert space

Classical and quantum information dynamics in terms of information flow in phase space

Models

- To begin with, consider linear dynamics (few coupled harmonic oscillators):
- Sufficiently simple to allow for comprehensive analytical solutions to test concepts and methods
- Sufficiently rich to exhibit incipient phenomena of exchange and reshuffling of information
- Continuous variables (∞-dimensional Hilbert space) permit a well-defined classical limit

Model

Hamiltonian: two coupled harmonic oscillators

$$H(\vec{r}_1,\vec{r}_2) = \sum_{i=1,2}\left(\frac{p_i^2}{2m} + \frac{m\omega_i^2 q_i^2}{2}\right) + V_{12}(q_1,q_2),$$

with position-position coupling,

$$V_{12}(q_1,q_2) = mcq_1q_2,$$

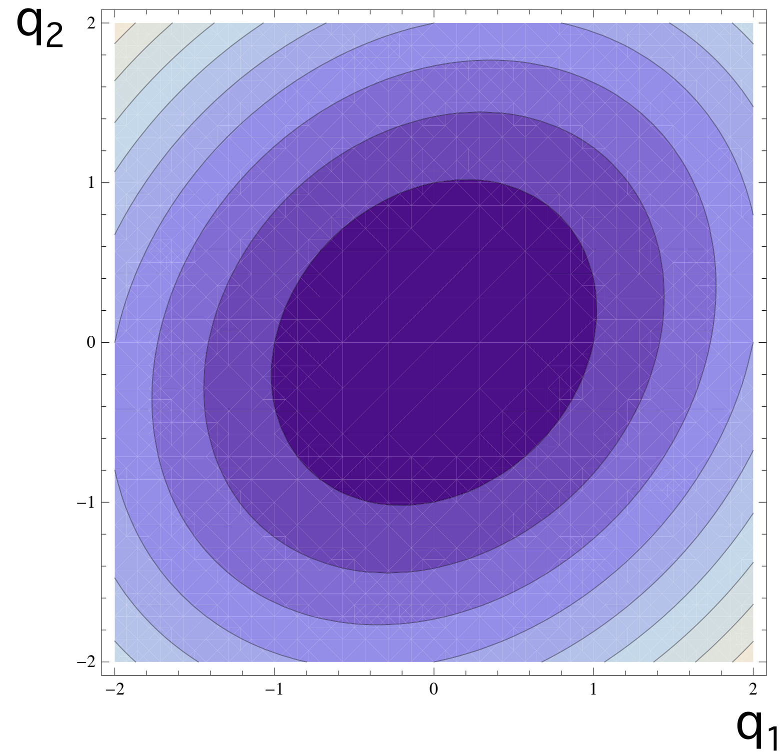
at resonance,

$$\omega_1 = \omega_2 =: \omega,$$

in the limit of weak coupling,

$$c << \omega^2.$$

potential (colour code)



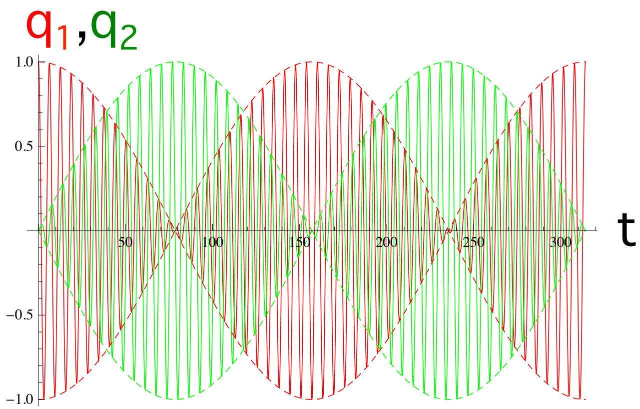
Basic dynamics

Eigenmodes, + in phase (> – >), – in counterphase (< – >):

$$\begin{pmatrix} q_+ \\ q_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \ddot{q}_\pm = -\omega_\pm^2 q_\pm, \qquad \omega_\pm = \sqrt{\omega^2 \pm c}$$

Principal phenomenon: beats with frequency $\omega_b^2 = (\omega_+^2 - \omega_-^2)/2 = c$

$$\begin{pmatrix} p_1(t) \\ q_1(t) \\ p_2(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t)\cos(\omega_b t) & -m\omega\sin(\omega t)\cos(\omega_b t) & \sin(\omega t)\sin(\omega_b t) & -m\omega\cos(\omega t)\sin(\omega_b t) \\ \frac{1}{m\omega}\sin(\omega t)\cos(\omega_b t) & \cos(\omega t)\cos(\omega_b t) & \frac{1}{m\omega}\cos(\omega t)\sin(\omega_b t) & \sin(\omega t)\sin(\omega_b t) \\ \sin(\omega t)\sin(\omega_b t) & -m\omega\cos(\omega t)\sin(\omega_b t) & \cos(\omega t)\cos(\omega_b t) & -m\omega\sin(\omega t)\cos(\omega_b t) \\ \frac{1}{m\omega}\cos(\omega t)\sin(\omega_b t) & \sin(\omega t)\sin(\omega_b t) & \frac{1}{m\omega}\sin(\omega t)\cos(\omega_b t) & \cos(\omega t)\cos(\omega_b t) \end{pmatrix} \begin{pmatrix} p_1(0) \\ q_1(0) \\ p_2(0) \\ q_2(0) \end{pmatrix},$$



Dynamics of phase-space probability density

Classical probability density distribution

$$\rho(\mathbf{r},t),\mathbf{r}=(\mathbf{r}_1,\mathbf{r}_2),\mathbf{r}_i=(p_i,q_i),i=1,2,\quad \int \mathrm{d}^4 r\rho(\mathbf{r})=1$$

Time evolution

$$\rho(\mathbf{r}'',t'')=\int \mathrm{d}^4 r'\rho(\mathbf{r}',t')G(\mathbf{r}'',t'';\mathbf{r}',t'),$$

Liouville propagator

$$G(\mathbf{r}'',t'';\mathbf{r}',t')=\delta\left(\mathbf{r}'-\mathrm{T}^{-1}(t'',t')\mathbf{r}''\right),$$

distribution moves with the classical Hamiltonian phase-space flow **r**(t), that is, transforms canonically as

$$\rho(\mathbf{r}'',t')=\rho\left(\mathbf{r}',t''\right),\mathbf{r}'\rightarrow\mathbf{r}''=T(t'',t')\mathbf{r}'$$

Dynamics of phase-space probability density

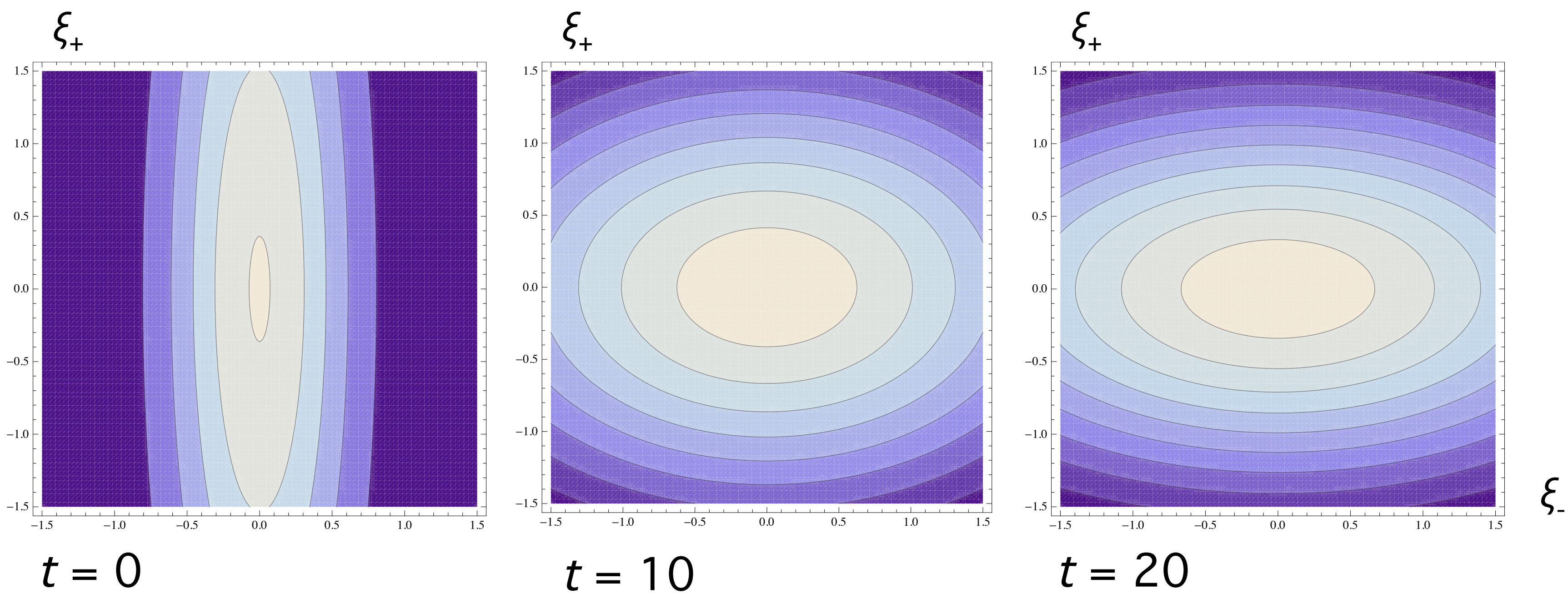
Gaussian initial distribution, $\boldsymbol{\rho}_i = (\eta_i, \xi_i)$, $\eta_i = (m\omega)^{-1/2} p_i$, $\xi_i = (m\omega)^{1/2} q_i$

$$\rho(\boldsymbol{\rho},0) = \rho_1(\boldsymbol{\rho}_1,0)\rho_2(\boldsymbol{\rho}_2,0), \rho_i(\boldsymbol{\rho}_i,0) = \frac{1}{2\pi S_i}\exp\left(\frac{-\boldsymbol{\rho}_i^2}{S_i}\right), i = 1,2$$

Time evolution

$$\rho(\boldsymbol{\rho},t) = \rho_1(\boldsymbol{\rho}_1,t)\rho_2(\boldsymbol{\rho}_2,t)\exp\left(\frac{-1}{4S_{12}(t)}\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2\right), \rho_i(\boldsymbol{\rho}_i,t) = \frac{1}{2\pi S_i}\exp\left(\frac{-\boldsymbol{\rho}_i^2}{2}\left(\frac{(\cos(\omega_b t))^2}{S_i} - \frac{(\sin(\omega_b t))^2}{S_{3-i}}\right)\right),$$

$$S_i(t) = S_1 S_2 / \left(S_{3-i}(\cos(\omega_b t))^2 + S_i(\sin(\omega_b t))^2\right), i = 1,2, S_{12}(t) = S_1 S_2 / (S_2 - S_1)\sin(2\omega_b t)$$



Dynamics of phase-space probability density

Partial distributions

$$\bar{\rho}_i(\boldsymbol{\rho}_i,t)=\int \mathrm{d}\rho_{3-i}^2\,\rho(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,t)=\frac{1}{2\pi S_i(t)}\exp\left(\frac{-\boldsymbol{\rho}_i^2}{2S_i(t)}\right), S_i(t)=S_i\left(\cos(\omega_\mathrm{b}t)\right)^2+S_{3-i}\left(\sin(\omega_\mathrm{b}t)\right)^2$$

Phase-space density factorizes, however not into partial distributions,

$$\rho(\boldsymbol{\rho},t)=\rho_+(\boldsymbol{\rho}_+,t)\rho_-(\boldsymbol{\rho}_-,t),$$

$$\rho_+(\eta_1,\xi_2,t)=\frac{1}{2\pi S_+(t)}\exp\left(-\frac{\eta_1^2}{2S_1}-\frac{\xi_2^2}{2S_2}+\frac{1}{4S_{12}(t)}\eta_1\xi_2\right),\rho_-(\eta_2,\xi_1,t)=\frac{1}{2\pi S_-(t)}\exp\left(-\frac{\eta_2^2}{2S_2}-\frac{\xi_1^2}{2S_1}-\frac{1}{4S_{12}(t)}\eta_2\xi_1\right)$$

and diagonalizes, but mixing subsystems in a time-dependent manner,

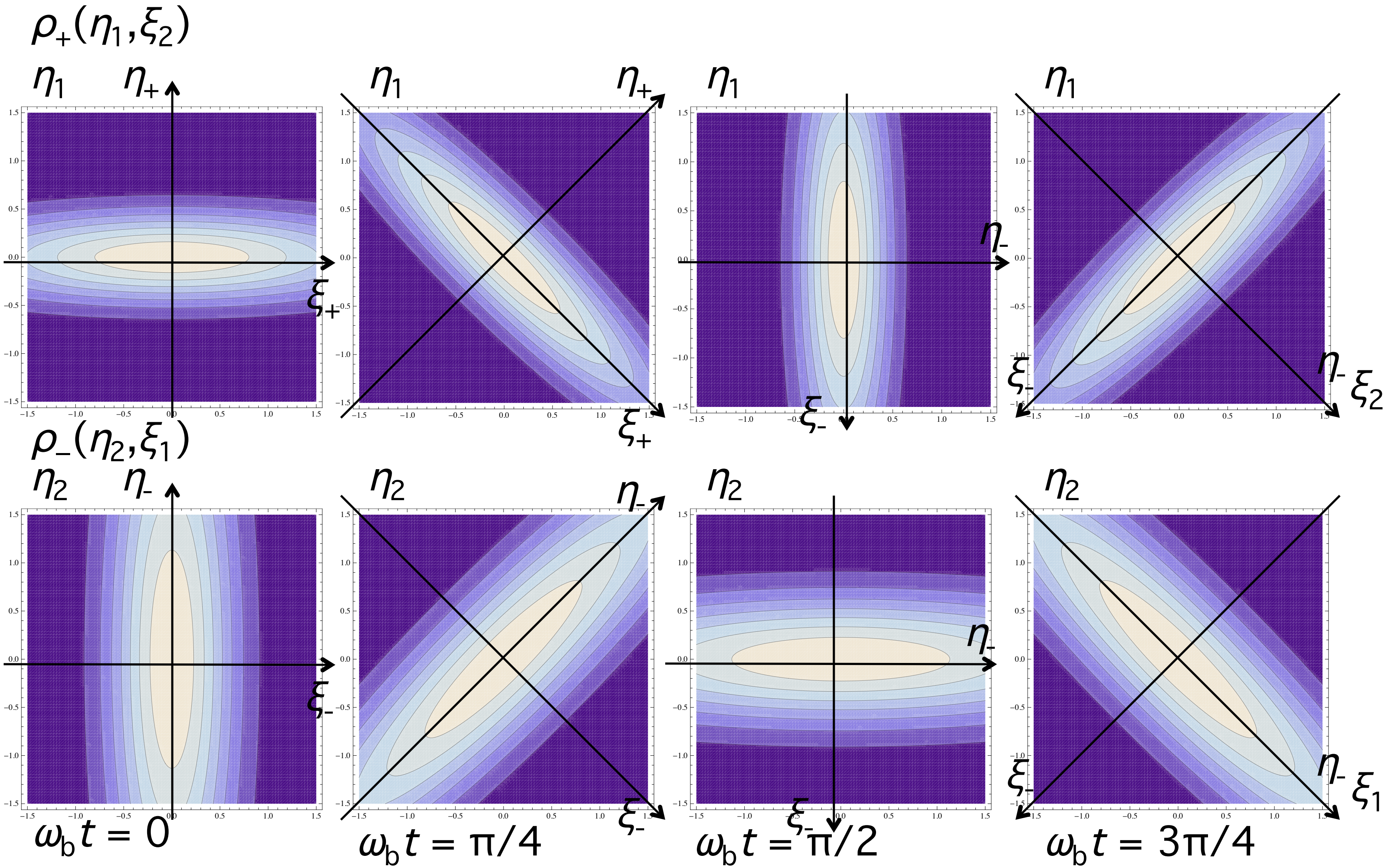
$$\rho_\pm\left(\boldsymbol{\rho}_\pm(t)\right)=\frac{1}{2\pi\sqrt{S_1S_2}}\exp\left(-\frac{\left(\rho_{\pm,1}(t)\right)^2}{2S_1}-\frac{\left(\rho_{\pm,2}(t)\right)^2}{2S_2}\right),$$

$$\begin{pmatrix}\rho_{-,1}(t)\\ \rho_{-,2}(t)\\ \rho_{+,1}(t)\\ \rho_{+,2}(t)\end{pmatrix}=\begin{pmatrix}\eta_-(t)\\ \xi_-(t)\\ \xi_+(t)\\ \eta_+(t)\end{pmatrix}=\begin{pmatrix}\cos(\omega_\mathrm{b}t) & 0 & 0 & -\sin(\omega_\mathrm{b}t)\\ \sin(\omega_\mathrm{b}t) & 0 & 0 & \cos(\omega_\mathrm{b}t)\\ 0 & \cos(\omega_\mathrm{b}t) & \sin(\omega_\mathrm{b}t) & 0\\ 0 & -\sin(\omega_\mathrm{b}t) & \cos(\omega_\mathrm{b}t) & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ -\xi_1\\ \eta_2\\ -\xi_2\end{pmatrix}$$

- Rigid Gaussian distribution rotates in four-dimensional phase space involving both subsystems.

Dynamics of phase-space probability density

- Rigid Gaussian distribution rotates in four-dimensional phase space involving both subsystems.



Information dynamics

Classical Shannon information

$$I_{\mathrm{cl}}(t) := -k \int \mathrm{d}r^{2f} \rho(\mathbf{r},t) \ln\left(\rho(\mathbf{r},t)\right)$$

is conserved under canonical transformations **r'** –> **r''** = T**r'**

$$-k \int \mathrm{d}r'^{2f} \rho(\mathbf{r}') \ln\left(\rho(\mathbf{r}')\right) = -k \int \mathrm{d}r''^{2f} \rho(\mathbf{r}'') \ln\left(\rho(\mathbf{r}'')\right)$$

Quantum von-Neumann information

$$I_{\mathrm{qm}}(t) := -k \,\mathrm{tr}\left[\hat{\rho} \ln\left(\hat{\rho}\right)\right]$$

is conserved under unitary transformations ρ' –> ρ'' = $U\rho' U^{+}$

$$-k \,\mathrm{tr}\left[\hat{\rho}' \ln\left(\hat{\rho}'\right)\right] = -k \,\mathrm{tr}\left[\hat{\rho}'' \ln\left(\hat{\rho}''\right)\right]$$

Classical information dynamics

Total information for a pair of coupled harmonic oscillators, Δ_i minimum resolvable phase-space cell for subsystem i, e.g., $\Delta_i = 2\pi\hbar$

$$I_{\text{tot}} = -k\left(\ln\left(\frac{\Delta_1\Delta_2}{4\pi^2 S_1 S_2}\right) - 2\right)$$

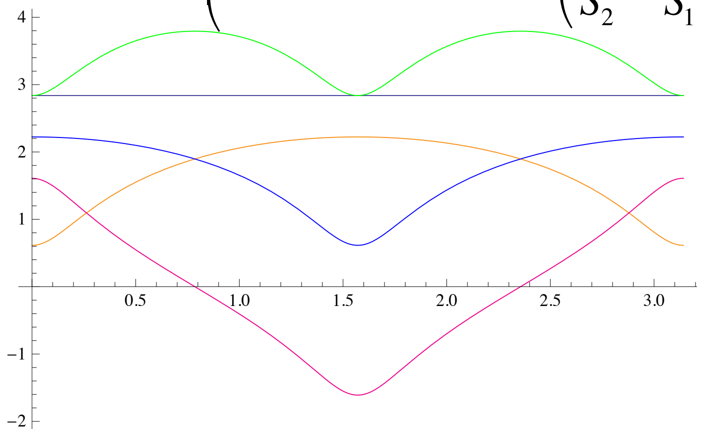
Partial information

$$I_i(t) = -k\int \mathrm{d}r_i^2\, \rho_i(\mathbf{r}_i,t)\ln\left(\rho_i(\mathbf{r}_i,t)\right) = -k\left(\ln\left(\frac{\Delta_i}{2\pi S_i(t)}\right) - 1\right), i = 1,2$$

Shared information (semiquantitative measure of inseparability)

$$I_{12}(t) = I_1(t) + I_2(t) - I_{\text{tot}} = k\ln\left(\frac{S_1(t)S_2(t)}{S_1 S_2}\right)$$

$$= k\ln\left(\left(\cos(\omega_b t)\sin(\omega_b t)\right)^2\left(\frac{S_1}{S_2} + \frac{S_2}{S_1}\right) + \left(\cos(\omega_b t)\right)^4 + \left(\sin(\omega_b t)\right)^4\right) > 0$$



sum of partial infos: $I_1 + I_2$
total info: I_{tot}

partial info subsystem 1: I_1
partial info subsystem 2: I_2

shared info: $I_1 + I_2 - I_{\text{tot}}$
difference of partial infos: $I_2 - I_1$

Classical information dynamics

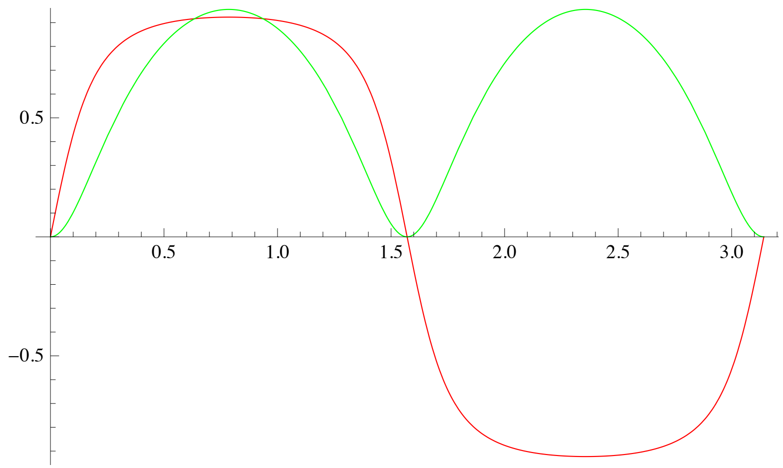
Correlation coefficient of phase-space flow

$$C(\mathbf{r}_1,\mathbf{r}_2,t) = \frac{\mathrm{Covar}\left(\mathbf{r}_1(t),\mathbf{r}_2(t)\right)}{\sqrt{\mathrm{Var}\left(\mathbf{r}_1(t)\right)\mathrm{Var}\left(\mathbf{r}_2(t)\right)}} = \frac{\left\langle \mathbf{r}_1(t)\times\mathbf{r}_2(t)\right\rangle}{\sqrt{\left\langle\left|\mathbf{r}_1(t)\right|^2\right\rangle\left\langle\left|\mathbf{r}_2(t)\right|^2\right\rangle}}$$

$$= \frac{\cos(\omega_\mathrm{b}t)\sin(\omega_\mathrm{b}t)\left(S_1^{-1}-S_2^{-1}\right)}{\sqrt{\left(S_1^{-1}\left(\cos(\omega_\mathrm{b}t)\right)^2+S_2^{-1}\left(\sin(\omega_\mathrm{b}t)\right)^2\right)\left(S_1^{-1}\left(\sin(\omega_\mathrm{b}t)\right)^2+S_2^{-1}\left(\cos(\omega_\mathrm{b}t)\right)^2\right)}}$$

Correlation coefficient vs. shared information

$C(r_1,r_2,t)$ $I_{12}(t)$



$\omega_\mathrm{b} t$

- Zeros of the shared information coincide with sign changes of the cross-correlation between the subsystems

Quantum dynamics

Relating quantum time evolution to classical phase space dynamics

- For harmonic Hamiltonians (up to quadratic in positions and momenta) the semiclassical approximation is exact.
- Quantum time evolution reduces to propagating the Wigner function with the classical phase-space flow.
- Quantum effects, if any, are restricted to the initial condition.

The Wigner function

WIGNER'S IDEA (1932)

DEFINE A REPRESENTATION OF THE DENSITY OPERATOR IN PHASE SPACE IN SUCH A WAY THAT ENSEMBLE AVERAGES CAN BE PERFORMED AS IN CLASSICAL STATISTICAL MECHANICS, I.E., AS PHASE-SPACE INTEGRALS.

SOLUTION (FROM POSITION REPRESENTATION, f FREEDOMS)

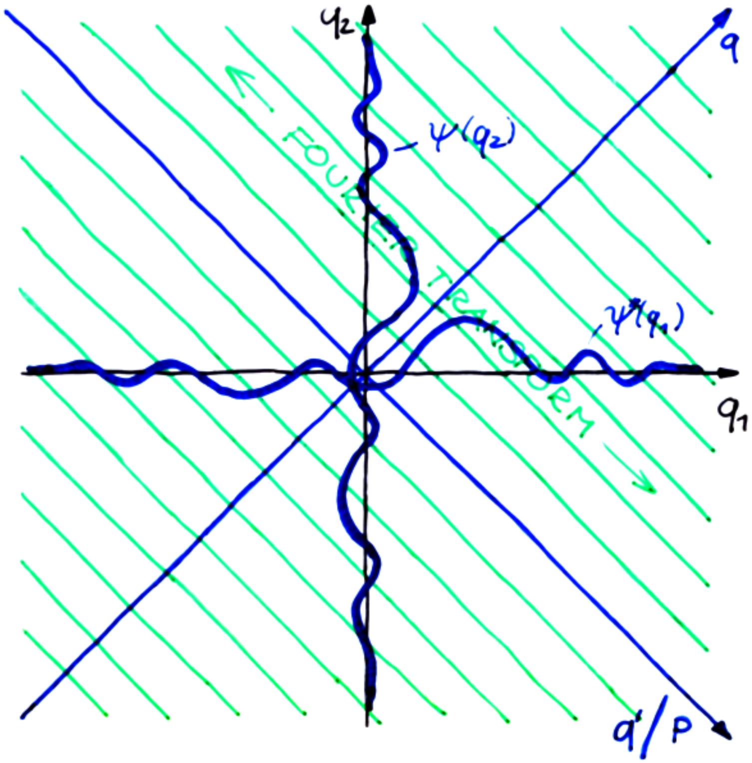
$$W(\vec{p},\vec{q}) := \frac{1}{h^f} \int d^f q' \exp\left(\frac{-i}{\hbar}\vec{p}\cdot\vec{q}'\right) \left\langle \vec{q}+\frac{\vec{q}'}{2} \right| \hat{\varrho} \left| \vec{q}-\frac{\vec{q}'}{2} \right\rangle$$

The Wigner function

FOR A PURE STATE $\hat{\varrho} = |\psi\rangle\langle\psi|$

$$W(\vec{p},\vec{q}) = \frac{1}{h^f} \int d^f q' \exp\left(\frac{i}{\hbar}\vec{p}\cdot\vec{q}'\right) \psi^*\underbrace{\left(\vec{q}-\frac{\vec{q}'}{2}\right)}_{q_2}\psi\underbrace{\left(\vec{q}+\frac{\vec{q}'}{2}\right)}_{q_1}$$

CALCULATION SCHEME



Quantum dynamics

Procedure to time-evolve the density operator

initial state $\rho(t')$

final state $\rho(t'')$

superposition of coherent states

↓ represent

density operator in position representation

$\langle \mathbf{q}_1|\rho(t')|\mathbf{q}_2\rangle$

↓ Weyl transform

Wigner function

$W(\mathbf{r}',t')$

→ propagate classically

Wigner function

$W(\mathbf{r}'',t'')$

↑ inverse Weyl transform

density operator in position representation

$\langle \mathbf{q}_1|\rho(t'')|\mathbf{q}_2\rangle$

↑ expand

superposition of coherent states

Quantum dynamics

Procedure to evaluate von-Neumann information

initial state $\rho(t')$

information measures

superposition of harmonic-oscillator eigenstates

expand

superposition of coherent states

represent

density operator in position representation

$<\mathbf{q}_1|\rho(t')|\mathbf{q}_2>$

Weyl transform

Wigner function

W($\mathbf{r}$',t')

final state $\rho(t'')$

information measures

superposition of harmonic-oscillator eigenstates

expand

superposition of coherent states

expand

density operator in position representation

$<\mathbf{q}_1|\rho(t'')|\mathbf{q}_2>$

inverse Weyl transform

Wigner function

W($\mathbf{r}$'',t'')

propagate classically

Quantum dynamics

Initial state: incoherent Gaussian superposition of coherent states

$$\hat{\rho}(0)=\hat{\rho}_1(0)\hat{\rho}_2(0),\hat{\rho}_i(0)=\frac{1}{2\pi S_i}\int d{r_i^{\gamma}}^2 \exp\left(\frac{-\left|\mathbf{r}_i^{\gamma}\right|^2}{S_i}\right)\left|\gamma_i\right\rangle\left\langle\gamma_i\right|, i=1,2$$

$$\left|\gamma_i\right\rangle=\exp\left(\frac{-\mathrm{i}}{\hbar}\left(\sqrt{\frac{2\hbar}{m\omega}}\mathrm{Re}(\gamma_i)\hat{p}_i-\sqrt{2m\hbar\omega}\,\mathrm{Im}(\gamma_i)\hat{q}_i\right)\right),\gamma_i=\sqrt{\frac{m\omega}{2\hbar}}q_i^{\gamma}+\frac{\mathrm{i}}{\sqrt{2m\hbar\omega}}p_i^{\gamma}$$

- closest possible quantum approximation to the classical initial phase-space density distribution:
- keeps the shape of the classical Gaussian, but
- respects quantum uncertainty for vanishing classical widths by assuming a "quantum broadening" by ħ/2 per degree of freedom.

Initial Wigner function

$$\rho_{\mathrm{W}}(\mathbf{r},0)=\rho_{\mathrm{W}1}(\mathbf{r}_1,0)\rho_{\mathrm{W}2}(\mathbf{r}_2,0),\rho_{\mathrm{W}i}(\mathbf{r}_i,0)=\frac{1}{2\pi S_i'}\exp\left(\frac{-1}{2S_i'}\left(m\omega q_i^2+\frac{p_i^2}{m\omega}\right)\right),S_i':=S_i+\frac{\hbar}{2}, i=1,2$$

Quantum dynamics

Time evolution of the Wigner function follows the classical phase-space flow, preserving the “quantum broadening”

$$\rho_{\mathrm{W}}(\mathbf{r},t)=\rho_{\mathrm{W}1}(\mathbf{r}_1,t)\rho_{\mathrm{W}2}(\mathbf{r}_2,t)\exp\left(\frac{-1}{2S_{12}(t)}\mathbf{r}_1\times\mathbf{r}_2\right),\rho_{\mathrm{W}i}(\mathbf{r}_i,t)=\frac{1}{2\pi S_i'}\exp\left(\frac{-1}{2S_i'(t)}\left(m\omega q_i^2+\frac{p_i^2}{m\omega}\right)\right),$$

$$S_i'(t)=S_1'S_2'/\left(S_{3-i}'\left(\cos(\omega_{\mathrm{b}}t)\right)^2+S_i'\left(\sin(\omega_{\mathrm{b}}t)\right)^2\right),i=1,2,$$

Time-evolved partial Wigner functions

$$\bar{\rho}_{\mathrm{W}i}(\mathbf{r}_i,t)=\int \mathrm{d}r_{3-i}^2\rho_{\mathrm{W}}(\mathbf{r},t)=\frac{1}{2\pi S_i'(t)}\exp\left(\frac{-1}{2S_i'(t)}\left(m\omega q_i^2+\frac{p_i^2}{m\omega}\right)\right),i=1,2,$$

in position representation

$$\left\langle Q_i-\frac{q_i}{2}\middle|\hat{\rho}_i(t)\middle|Q_i-\frac{q_i}{2}\right\rangle=\sqrt{\frac{m\omega}{2\pi S_i'(t)}}\exp\left(\frac{-m\omega}{2S_i'(t)}\left(Q_i^2+\frac{\left(S_i'(t)\right)^2}{\hbar^2}q_i^2\right)\right),$$

Quantum dynamics

Time-evolved partial distribution in terms of coherent states

$$\hat{\rho}_i(t) = \frac{\hbar}{\pi S_i(t)} \exp\left(\frac{-\hbar}{S_i(t)} |\gamma_i|^2 \right) |\gamma_i\rangle_i \langle\gamma_i|_i, i = 1,2,$$

- diagonalizes the density operator, but in an overcomplete, hence non-unique, representation

Time-evolved partial distribution in terms of harmonic-oscillator number states *H* |*n*> = E_n |*n*>

$$\hat{\rho}_i(t) = \frac{\hbar}{S_i(t)} \sum_{n_i=0}^{\infty} \left(1 + \frac{\hbar}{S_i(t)} \right) |n_i\rangle_i \langle n_i|_i, i = 1,2,$$

- diagonalizes the density operator in a complete and unique representation

- allows to evaluate expectation values such as information measures as eigenvalue functions

Quantum dynamics

The Wigner function factorizes at any time

$$\rho_{\mathrm{W}}(\mathbf{r},t)=\rho_{\mathrm{W}}^{1}\left(\boldsymbol{\rho}^{1}(t)\right)\rho_{\mathrm{W}}^{2}\left(\boldsymbol{\rho}^{2}(t)\right),\rho_{\mathrm{W}}^{i}\left(\boldsymbol{\rho}^{i}(t)\right)=\frac{1}{2\pi\sqrt{S_1'S_2'}}\exp\left(-\frac{\left(\rho_1^i(t)\right)^2}{2S_1'}-\frac{\left(\rho_2^i(t)\right)^2}{2S_2'}\right),i=1,2,$$

$$\begin{pmatrix}\rho_1^1(t)\\ \rho_2^1(t)\\ \rho_1^2(t)\\ \rho_2^2(t)\end{pmatrix}=\begin{pmatrix}\eta^1(t)\\ \xi^1(t)\\ \xi^2(t)\\ \eta^2(t)\end{pmatrix}=\begin{pmatrix}\cos(\omega_{\mathrm{b}}t) & 0 & 0 & -\sin(\omega_{\mathrm{b}}t)\\ \sin(\omega_{\mathrm{b}}t) & 0 & 0 & \cos(\omega_{\mathrm{b}}t)\\ 0 & \cos(\omega_{\mathrm{b}}t) & \sin(\omega_{\mathrm{b}}t) & 0\\ 0 & -\sin(\omega_{\mathrm{b}}t) & \cos(\omega_{\mathrm{b}}t) & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ -\xi_1\\ \eta_2\\ -\xi_2\end{pmatrix}$$

as does the density operator, e.g., in position representation

$$\hat{\rho}=\hat{\rho}^1\hat{\rho}^2,\left\langle Q^i(t)-\frac{q^i(t)}{2}\right|\hat{\rho}^i\left|Q^i(t)-\frac{q^i(t)}{2}\right\rangle=\sqrt{\frac{m\omega}{2\pi S_i'}}\exp\left(\frac{-m\omega}{2S_i'}\left(\left(Q^i(t)\right)^2+\frac{S_i'^2}{\hbar^2}\left(q^i(t)\right)^2\right)\right),$$

- factorizes distributions, but mixing subsystems in a time-dependent manner
- different widths $S_1 \neq S_2$ for position and momentum require to expand ρ in terms of *squeezed* coherent states

Quantum information dynamics: total information

Total information

$$I_{vN,\mathrm{tot}} = -k\sum_{i=1}^{2}\left(\ln\left(\frac{\hbar}{S_i}\right) - \left(\frac{S_i}{\hbar}+1\right)\ln\left(1+\frac{\hbar}{S_i}\right)\right)$$

Classical limit

$$I_{vN,\mathrm{tot}} \xrightarrow{\hbar/S_i<<1} -k\sum_{i=1}^{2}\left(\ln\left(\frac{\hbar}{S_i}\right)-1\right)$$

Quantum limit (limit of pure states)

$$I_{vN,\mathrm{tot}} = -k\sum_{i=1}^{2}\left(\ln\left(\frac{\hbar}{S_i}\right) - \left(1+\frac{S_i}{\hbar}\right)\ln\left(\frac{\hbar}{S_i}\left(1+\frac{S_i}{\hbar}\right)\right)\right) \xrightarrow{S_i/\hbar_i<<1} 0$$

Purity

$$\mathrm{tr}\left[\hat{\rho}^2\right] = \prod_{i=1}^{2}\left(1+\frac{S_i}{\hbar/2}\right)^{-1}$$

Quantum information dynamics: partial information

Partial information

$$I_{vN,i}(t) = -k\left(\ln\left(\frac{\hbar}{S_i(t)}\right) - \left(\frac{S_i(t)}{\hbar}+1\right)\ln\left(1+\frac{\hbar}{S_i(t)}\right)\right)$$

Classical limit

$$I_{vN,i}(t) \xrightarrow{\hbar/S_i(t)<<1} -k\sum_{i=1}^{2}\left(\ln\left(\frac{\hbar}{S_i(t)}\right)-1\right)$$

Quantum limit (pure-state limit)

$$I_{vN,i}(t) = -k\sum_{i=1}^{2}\left(\ln\left(\frac{\hbar}{S_i(t)}\right) - \left(1+\frac{S_i(t)}{\hbar}\right)\ln\left(\frac{\hbar}{S_i(t)}\left(1+\frac{S_i(t)}{\hbar}\right)\right)\right) \xrightarrow{S_i(t)/\hbar_i<<1} 0$$

Purity

$$\mathrm{tr}\left[\left(\hat{\rho}_i(t)\right)^2\right] = \left(1+\frac{S_i(t)}{\hbar/2}\right)^{-1}$$

Quantum inseparability (entanglement) criteria

How to assess entanglement in mixed states in an infinite-dimensional Hilbert space?

There exist criteria for separability vs. inseparability of quantum states that can be expressed as a Gaussian Wigner function, referring to the variance matrix of the Gaussian:

- Peres – Horodecki criterion, based on partial transpose of the density matrix (R. Simon, *Phys. Rev. Lett.* 84, 2726 (2000))
- Cirac – Zoller criterion, based on the minimum-uncertainty condition for the density operator (C. M. Duan, G. Giedke, J. I. Cirac, P. Zoller, *Phys. Rev. Lett.* 84, 2722 (2000))

Quantum inseparability (entanglement) criteria

Write Wigner function as generalized (skew) Gaussian

$$\rho_{\mathrm{W}}(\mathbf{r},t)=\frac{1}{4\pi^2 S_1'(t)S'(t)}\exp\left(\frac{-1}{2}\begin{pmatrix}\boldsymbol{\rho}_1\\ \boldsymbol{\rho}_2\end{pmatrix}^{t}\mathbf{M}_{\mathrm{W}}(t)\begin{pmatrix}\boldsymbol{\rho}_1\\ \boldsymbol{\rho}_2\end{pmatrix}\right),\mathbf{M}_{\mathrm{W}}(t)=\begin{pmatrix}1/S_1'(t) & 0 & 0 & -1/S_{12}'(t)\\ 0 & 1/S_1'(t) & 1/S_{12}'(t) & 0\\ 0 & 1/S_{12}'(t) & 1/S_2'(t) & 0\\ -1/S_{12}'(t) & 0 & 0 & 1/S_2'(t)\end{pmatrix}$$

Variances

$$\langle q_1^2\rangle = S_1''(t)/m\omega,\quad \langle q_1 p_1\rangle = 0,\quad \langle q_1 q_2\rangle = 0,\quad \langle q_1 p_2\rangle = -S_{12}(t),$$

$$\langle p_1 q_1\rangle = 0,\quad \langle p_1^2\rangle = m\omega S_1''(t),\quad \langle p_1 q_2\rangle = S_{12}(t),\quad \langle p_1 p_2\rangle = 0,$$

$$\langle q_2 q_1\rangle = 0,\quad \langle q_2 p_1\rangle = S_{12}(t),\quad \langle q_2^2\rangle = S_2''(t)/m\omega,\quad \langle q_2 p_2\rangle = 0,$$

$$\langle p_2 q_1\rangle = -S_{12}(t),\quad \langle p_2 p_1\rangle = 0,\quad \langle p_2 q_2\rangle = 0,\quad \langle p_2^2\rangle = m\omega S_1''(t),$$

$$S_i''(t) = S_i(t)+\hbar, i=1,2,$$

Quantum inseparability (entanglement) criteria: Cirac – Zoller criterion

For EPR-like operators

$$\hat{u} := \sqrt{m\omega}\left(|a|\hat{q}_1 + \frac{1}{a}\hat{q}_2\right), \hat{v} := \frac{1}{\sqrt{m\omega}}\left(|a|\hat{p}_1 + \frac{1}{a}\hat{p}_2\right),$$

find variances

$$\left\langle(\Delta\hat{u})^2\right\rangle = \left\langle(\Delta\hat{v})^2\right\rangle = S_1\left(a^2(\cos(\omega_b t))^2 + a^{-2}(\sin(\omega_b t))^2\right) +$$

$$+S_2\left(a^2(\sin(\omega_b t))^2 + a^{-2}(\cos(\omega_b t))^2\right) + \hbar\left(a^2 + a^{-2}\right),$$

check

$$\left\langle(\Delta\hat{u})^2\right\rangle + \left\langle(\Delta\hat{v})^2\right\rangle = 2S_1\left(a^2(\cos(\omega_b t))^2 + a^{-2}(\sin(\omega_b t))^2\right) +$$

$$+2S_2\left(a^2(\sin(\omega_b t))^2 + a^{-2}(\cos(\omega_b t))^2\right) + 2\hbar\left(a^2 + a^{-2}\right)$$

$$\geq a^2 + a^{-2},$$

for all values of $S_1, S_2 \geq 0$, as necessary and sufficient criterion for separability (non-entanglement).

Quantum inseparability (entanglement) criteria: Peres – Horodecki criterion

Write variance matrix as square matrix of (2×2)-blocks

$$V = \hbar^{-1}\begin{pmatrix} A & C \\ C^{\mathrm{t}} & B \end{pmatrix},$$

$$A = \begin{pmatrix} m\omega & 0 \\ 0 & 1/m\omega \end{pmatrix} S_1''(t), B = \begin{pmatrix} m\omega & 0 \\ 0 & 1/m\omega \end{pmatrix} S_2''(t), C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S_{12}(t),$$

check

$$\det(A)\det(B) + \left(\frac{\hbar^2}{4} - \det(C)\right) - \mathrm{tr}(AJCJBJC^{\mathrm{t}}J) = \left(S_1''S_2'' - \frac{\hbar^2}{4}\right)^2 + \frac{\hbar^2}{4} S_1''S_2'' \geq 0$$

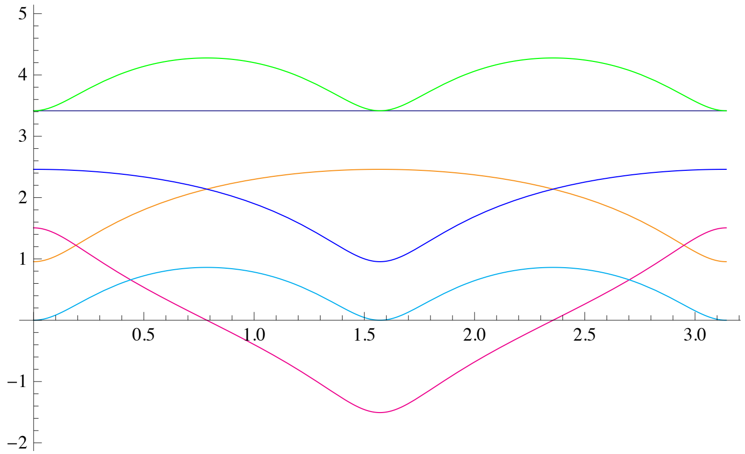
for all values of $S_1, S_2 \geq 0$, as necessary and sufficient criterion for separability (non-entanglement).

Quantum inseparability (entanglement) criteria

- Both criteria indicate that the system is permanently in a separable (non-entangled) state
- Indeed, the density operator factorizes for all times –
- however, not into the constituent subsystems.
- Subsystems do get entangled at all times except for the nodes of the classical beats at $t = n\pi/\omega$, n integer.

Quantum information dynamics

Information measures for $\hbar = 0.1$

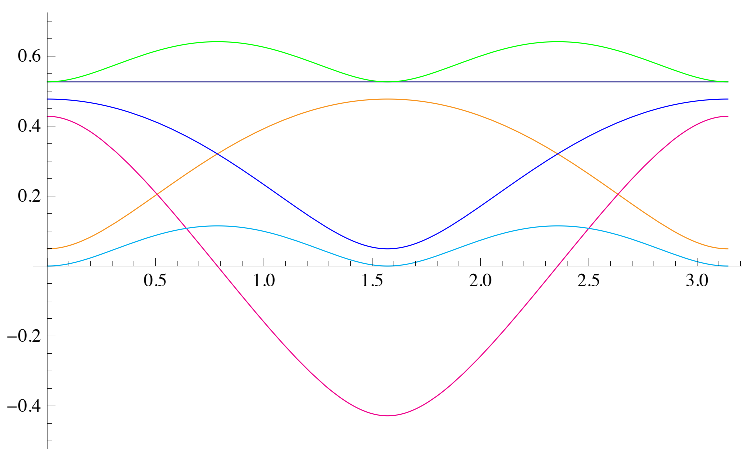


sum of partial infos: $I_1 + I_2$
total info: I_{tot}

partial info subsystem 1: I_1
partial info subsystem 2: I_2

shared info: $I_1 + I_2 - I_{tot}$
difference of partial infos: $I_2 - I_1$

Information measures for $\hbar = 10$



sum of partial infos: $I_1 + I_2$
total info: I_{tot}

partial info subsystem 1: I_1
partial info subsystem 2: I_2

shared info: $I_1 + I_2 - I_{tot}$
difference of partial infos: $I_2 - I_1$

Conclusions

- Implemented a simple bipartite system of weakly coupled harmonic oscillators at resonance.
- Studied its time evolution on basis of the classical phase-space density and quantum Wigner function dynamics.
- Beats induce a periodic exchange of energy and swapping of information between the subsystems.
- Zeros of shared information (“sudden deaths of entanglement”) coincide with sign changes of the crosscorrelation between subsystems.

Challenges for future work

- Study “less trivial” initial preparations, e.g., pure yet entangled states

$$\hat{\rho} = |\Psi\rangle\langle\Psi|, |\Psi\rangle = \frac{1}{\sqrt{2}}\left(|n'\rangle_1 |n''\rangle_2 + |n''\rangle_1 |n'\rangle_2\right)$$

or

$$|\Psi\rangle = \frac{1}{2\pi\sqrt{\sigma_1\sigma_2}} \iint d\gamma_1^2 d\gamma_2^2 \exp\left(\frac{-|\gamma_1|^2}{2\sigma_1} - \frac{-|\gamma_2|^2}{2\sigma_2}\right) |\gamma_1\rangle_1 |\gamma_2\rangle_2$$

- Generalize towards nonlinear dynamics

- Increase number of harmonic freedoms

- Define clssical and quantum phase-space information flows consistent with basic features such as quantum non-locality of information