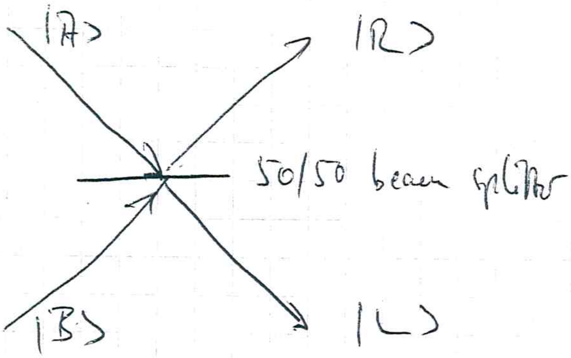
Ou & Mandel PRL 61 50 (1988)

Hong, Ou & Mandel PRL 59 2044 (1987)

C) Entanglement of (in-)distinguishable particles

C.1 The Hong-Ou-Mandel ~~effect~~ experiment



Fresnel's equations:

$$|A\rangle \to \frac{1}{\sqrt{2}}\left(|L\rangle + |R\rangle\right)$$
$$|B\rangle \to \frac{1}{\sqrt{2}}\left(|L\rangle - |R\rangle\right) \qquad (19)$$

Assume initial state

$$|\psi_{HOM,init}\rangle = \frac{1}{\sqrt{2}}\left[\,|\underbrace{A,\overset{\substack{\text{horizontal polarization}\\ \downarrow}}{H}}_{\text{photon 1}};\underbrace{B,V}_{\text{photon 2}}\rangle + |\underbrace{B,\overset{\substack{\text{vertical polarization}\\ \swarrow}}{V}}_{\text{photon 1}};\underbrace{A,H}_{\text{photon 2}}\rangle\,\right] \qquad (20)$$

After transmission through beam splitter

$$|\psi_{HOM,fin}\rangle = \frac{1}{2\sqrt{2}}\Big[\,|LHLV\rangle - |LHRV\rangle + |RHLV\rangle - |RHRV\rangle$$
$$+ |LVLH\rangle - |RVLH\rangle + |LVRH\rangle - |RVRH\rangle\,\Big] \qquad (21)$$

Post-selection of coincident events; i.e. discard events when only one detector clicks:

$$|\psi_{HOM,fin,coincident}\rangle \sim -|LHRV\rangle + |RHLV\rangle - |RVLH\rangle + |LVRH\rangle \qquad (22)$$

Observation:

While $|\Psi_{\text{HOM, init}}\rangle$ is not entangled in the polarization degree of freedom, $|\Psi_{\text{HOM, fin, coincident}}\rangle$ is maximally entangled!

Cause: – (maximally) ambiguous detector setting $|L\rangle$ and $|R\rangle$ instead of unambiguous setting $|A\rangle$ and $|B\rangle$

– symmetrization of initial two-photon wave function $\longrightarrow$ coherent superposition of two indistinguishable two-particle paths

C.2 Detector-level density matrix

Experimentally observed expectation values at detectors L and R are given by an observable of ~~the~~ the following structure:

$$O_d(\alpha,\beta) = \underbrace{O_L \otimes \alpha}_{\text{particle 1}} \otimes \underbrace{O_R \otimes \beta}_{\text{particle 2}} + \underbrace{O_R \otimes \beta}_{\text{particle 1}} \otimes \underbrace{O_L \otimes \alpha}_{\text{particle 2}}, \quad (23)$$

with the mutually orthogonal projectors O_L, O_R (on spatially localized detectors ~~O_L, O_R~~ L and R),

$$O_{L/R}^2 = O_{L/R}, \quad O^{\dagger}_{L/R} = O_{L/R}, \quad O^{\dagger}_{L/R} O_{L/R} = O_{L/R}, \quad (24)$$

and α, β observables on internal degrees of freedom of the scattered particles.

The density matrix of the ~~internal~~ two-particle state in the internal degrees of freedom, upon detection at the spatial detectors L and R, is obtained by state tomography, i.e. a complete set of observables for the internal degrees of freedom:

$$\underbrace{\rho_d}_{\text{detector-level density matrix}} = \text{const} \sum_{ij} \underbrace{\chi_i \otimes \chi_j}_{\text{complete set of observables on either particle}} \, tr\left(O_d(\chi_i, \chi_j)\, \underbrace{\rho}_{\text{prepared state}}\right) \qquad (25)$$

<u>Observation:</u> Entanglement measures applied on ρ_d (where particles are distinguishable through detection in L, R) quantify the entanglement measured between the detectors L and R! Result depends on prepared state ρ <u>and</u> on detector settings!

C.3 Path weights and effective ~~dist~~ indistinguishability

We saw in C.1 that two different two-particle paths interfere, due to the ambiguity induced by the beam splitter mappings (19). We therefore define the <u>path weights</u>

$$\begin{aligned} D_{LR} &= \langle A | O_L | A \rangle \langle B | O_R | B \rangle \\ D_{RL} &= \langle A | O_R | A \rangle \langle B | O_L | B \rangle \end{aligned} \qquad (26)$$

[illegible] $|\gamma| = |\langle A|O_L|B\rangle\langle B|O_R|A\rangle| =$

$$\overset{(24)}{=} |\langle A|O_L^\dagger O_L|B\rangle\langle B|O_R^\dagger O_R|A\rangle|$$

$$\le \| O_L|A\rangle\| \, \| O_L|B\rangle\| \, \| O_R|B\rangle\| \, \| O_R|A\rangle\|$$

Cauchy-Schwarz

$$\overset{(26)}{=} \sqrt{D_{LR} D_{RL}}$$

which quantify the probabilities that ~~the~~

$\overline{D_{LR}}$ the particle prepared in $|A\rangle$ is detected in $|L\rangle$ while the particle prepared in $|B\rangle$ is detected in $|R\rangle$

$\overline{D_{RL}}$ the particle prepared in $|A\rangle$ is detected in $|R\rangle$ while the particle prepared in $|B\rangle$ is detected in $|L\rangle$

The detector setting is unambiguous if one path weight vanishes, since ~~then the~~ a coincident event then unambiguously defines the provenance of the particle.

If both path weights are non-zero, which-way information is at least partially lost (and both two-particle paths ~~must~~ add up at least partially coherently).

A quantifier for the coherence between the two two-particle paths is the <u>effective ~~ds~~ indistinguishability</u>

$$\mu := \langle A|O_L|B\rangle\langle B|O_R|A\rangle \qquad (27)$$

which in modulus is bounded by

$$0 \leq |\mu| \leq \sqrt{D_{LR}D_{RL}} =: \mu_{max} \qquad (28)$$

<u>Observationally</u>, $|\mu|$ quantifies how strongly which-way information is erased by the measurement: If $|\mu| = \mu_{max}$, then $O_{R/L}|B\rangle$ and $O_{R/L}|A\rangle$ are linearly dependent and cannot be discriminated by measurement.

2) The (anti-)symmetrization of the wave function has an impact on observables only if the path weights do both not vanish and if $\beta \neq 0$.

C.4 Example

Consider two spin-$\frac{1}{2}$ particles prepared in the external quantum states $|A\rangle$ and $|B\rangle$ with

$$\langle A | B \rangle = 0 , \tag{29}$$

in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \Big[\cos\varepsilon \, |A,\uparrow; B,\downarrow\rangle + \sin\varepsilon \, |A,\downarrow; B,\uparrow\rangle + \delta \big[\cos\varepsilon \, |B,\downarrow; A,\uparrow\rangle + \sin\varepsilon \, |B,\uparrow; A,\downarrow\rangle \big] \Big], \tag{30}$$

where

$$\delta = \begin{cases} +1 & \text{for bosons} \\ -1 & \text{fermions} \end{cases} \tag{31}$$

and $\varepsilon \in \mathbb{R}$ controls the particles' entanglement in the spin-degree of freedom, in the prepared state, with concurrence

$$C_{|\psi\rangle}(\varepsilon) \overset{(!)}{=} 2 \, |\cos\varepsilon \sin\varepsilon| \tag{32}$$

($\varepsilon = 0, \frac{\pi}{2} \rightarrow$ no entanglement;
$\varepsilon = \frac{\pi}{4} \rightarrow$ maximally entangled)

C.4.1 Squared distance of particles

$$\langle \psi | (x_1 \otimes \mathbb{1}_2 - \mathbb{1}_1 \otimes x_2)^2 | \psi \rangle$$

$$\underset{(30)}{=} \langle A | x^2 | A \rangle + \langle B | x^2 | B \rangle - 2 \langle A | x | A \rangle \langle B | x | B \rangle$$

$$- 4 \delta \cos\varepsilon \sin\varepsilon \, |\langle A | x | B \rangle|^2 \qquad (33)$$

~~First~~ Second line: "classical" result for distinguishable particles

Third line: "exchange interaction" term controlled by

- spatial overlap $\langle A | x | B \rangle$
- statistics (δ)
- entanglement of prepared state (by tuning ε, can make bosons behave like fermions and vice versa)

C.4.2 Coincidence detection rate

$$T = \langle \psi | O_d(\mathbb{1}, \mathbb{1}) | \psi \rangle$$

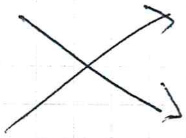
$$\underset{(23,30)}{=} D_{LR} + D_{RL} + 4 \delta \cos\varepsilon \sin\varepsilon \, \mathrm{Re}(f) \qquad (34)$$

HOM-setting $\Leftrightarrow D_{LR} = D_{RL} = \frac{1}{4}$, $f = -f_{max}$ $\Rightarrow$

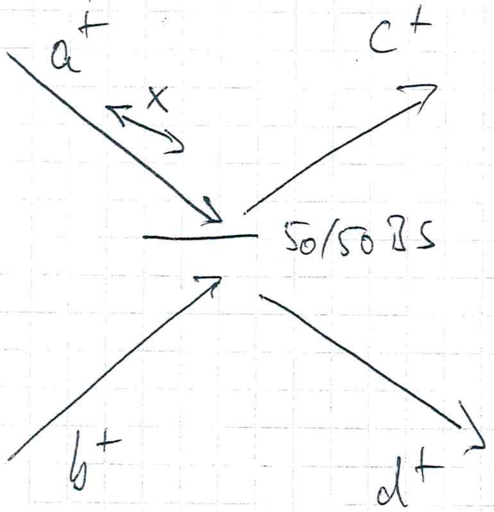
$$\underset{\delta=1}{\Rightarrow} \quad T = \frac{1}{2} - \cos\varepsilon \sin\varepsilon = \begin{cases} 0 & \text{for } \varepsilon = \pi/4 \quad \underline{\text{bunching}} \\ 1 & \text{for } \varepsilon = \frac{3\pi}{4} \quad \underline{\text{antibunching}} \end{cases}$$

~~$\Rightarrow$ Constructive / destructive interference~~

$\longleftrightarrow$ constructive ($\varepsilon = \frac{3\pi}{4}$) / destructive ($\varepsilon = \frac{\pi}{4}$) interference of two-particle paths


and


C.5 Four particle scattering – two particles per mode



x = path delay between photons in modes a and b

$\updownarrow$

temporal delay of photons at BS

C.5.1 Photon (in-)distinguishability modes

photon with arrival time t_j at BS:

$$A^+_{t_j} |0\rangle = |1_{t_j}\rangle = \int_{-\infty}^{+\infty} d\omega \, \frac{1}{\sqrt{\pi}\,\Delta\omega} \, e^{-\frac{(\omega-\omega_0)^2}{2\Delta\omega^2}} \, e^{i\omega t_j} A^+_\omega |0\rangle \tag{35}$$

distinguishability of photons with different arrival times t_1 and t_2 quantified by

$\cancel{\alpha = \langle t_1 | t_2 \rangle}$

$$\Leftrightarrow \quad \alpha = \langle 1_{t_1} | 1_{t_2} \rangle \tag{36}$$

Ergo, have Gram-Schmidt decomposition of $|1_{t_2}\rangle$ into

$$\alpha\,|1_{t_1}\rangle \quad \text{[indistinguishable component / mode]}$$

and

$$\sqrt{1-\alpha^2}\,|\tilde{1}_{t_1}\rangle \;, \quad |\tilde{1}_{t_1}\rangle = \frac{|1_{t_2}\rangle - \alpha|1_{t_1}\rangle}{1-\alpha^2}$$

[distinguishable component / mode],

i.e.

$$\cancel{A^+_{t_2}|0\rangle = |1_{t_2}\rangle = \alpha|1_{t_1}\rangle + \sqrt{1-\alpha^2}|1_{t_1}\rangle}$$

$$A^+_{t_2}|0\rangle = \alpha\,|1_{t_1}\rangle + \sqrt{1-\alpha^2}\,|\tilde{1}_{t_1}\rangle \qquad (37)$$

$\alpha \in \mathbb{R}$ (without loss of generality)

$$|\alpha|^2 = \exp\left[-\Delta\omega^2 (t_2-t_1)^2/2\right]$$

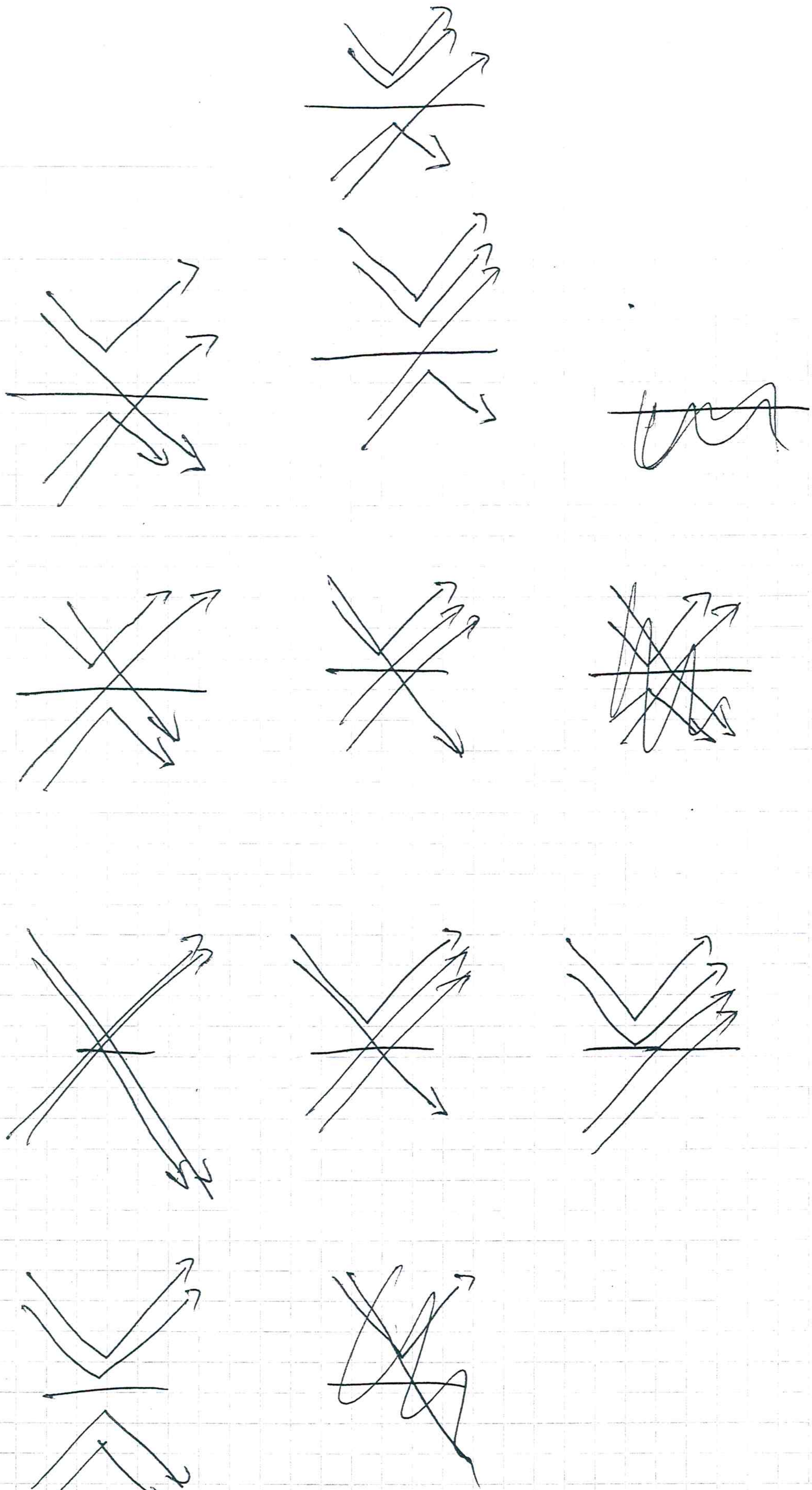
$$|\alpha|^2 = \begin{cases} 1 & \text{for strictly indistinguishable} \\ 0 & \text{for fully distinguishable} \end{cases} \quad \text{photons!}$$

C.5.2 HOM again – one photon per mode

Two-photon state given by

$$a^+_{t_1}\, \underset{\substack{\uparrow\\ \text{vacuum of modes } a \text{ and } b}}{b^+_{t_2}\,|0\rangle} \overset{(37)}{=} \alpha\,\underbrace{|1,1\rangle}_{\text{indistinguishable}} + \sqrt{1-\alpha^2}\,\underbrace{|1,\tilde{1}\rangle}_{\text{distinguishable}} \qquad (38)$$

mode a mode b



Only indistinguishable component $\sim \alpha$ will lead to bunching ~~out~~ events as predicted by (34), and ~~if~~ two-particle interference will fade away as $|\alpha|^2 \to 0$?!

C.5.3 Something new ...

The analogous Gram-Schmidt decomposition (37) for two photons in each mode produces

$$\frac{1}{2}(a^\dagger_{t_1})^2 (b^\dagger_{t_2})^2 |0\rangle = \alpha^2 |\overset{a}{2},\overset{b}{2}\rangle + \sqrt{2}\,\alpha\sqrt{1-\alpha^2}\,|\overset{a}{2},\underbrace{1,\tilde{1}}_{b}\rangle + (1-\alpha^2)|\underset{a}{2},\underset{b}{\tilde{2}}\rangle \qquad (39)$$

~~Here~~

Herein:

$|2,2\rangle \equiv$ strictly indistinguishable four particle paths

$|2,\tilde{2}\rangle \equiv$ fully distinguishable particles

NEW $\to$ $|2,1,\tilde{1}\rangle \equiv$ one photon in mode b strictly indistinguishable from photons in a, the other one fully distinguishable

$\to$ three-particle interference!

Probability to detect m particles in output c and n particles in output d, given $N/2$ particles in each a and b:

$$P^{(N,m,n)}(\alpha) = \sum_{\substack{\text{distinguishability} \\ \text{type}}} P^{(N,m,n)}_{\text{type}} \, W^{(N)}_{\text{type}}(\alpha) \qquad (40)$$

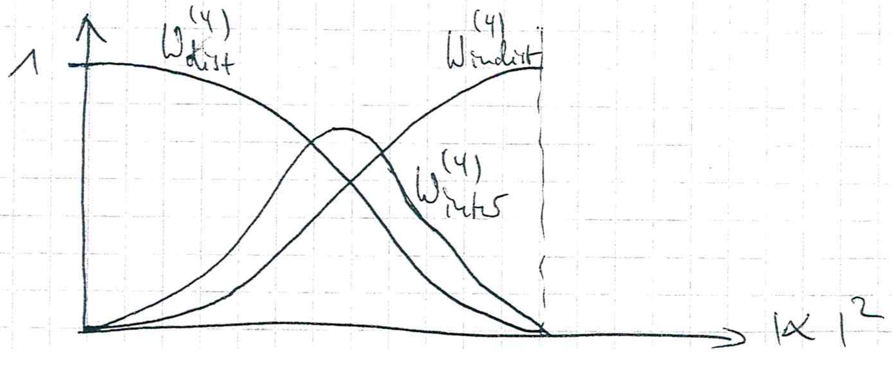
for $N \geq 4$: partially distinguishable produces cross terms alike (39)

with $W^{(N)}_{\text{type}}(\alpha) \equiv [\cancel{\text{Vorfakt}}\ \text{prefactors in } (38, 39)]^2$

$P^{(N,m,n)}_{\text{type}} \equiv$ detection probability of event (m,n) for given distinguishability type and

$a^+ \to \frac{1}{\sqrt{2}}(c^+ + d^+)$, $b^+ \to \frac{1}{\sqrt{2}}(c^+ - d^+)$

$P^{(4,m,n)}$	indist	inter	dist
(4,0)	3/8	3/16	1/6
(2,2)	1/4	1/8	3/8



→ While four-particle interference & ($W^{(4)}_{indist}$) fades out, three-particle interference ($W^{(3)}_{dist}$) kicks in !
~~before only distinguishi~~

→ $\boxed{\text{non-monotonous distinguishability-transition !!}}$

(→ Plots paper !!)

$\boxed{\text{distinguishability transition} \sim \text{quantum-to-classical transition}}$